

# Level statistics of one-dimensional Schrödinger operators with random decaying potential

Shinichi Kotani <sup>\*</sup>      Fumihiko Nakano <sup>†</sup>

October 17, 2012

## Abstract

We study the level statistics of one-dimensional Schrödinger operator with random potential decaying like  $x^{-\alpha}$  at infinity. We consider the point process  $\xi_L$  consisting of the rescaled eigenvalues and show that : (i)(ac spectrum case) for  $\alpha > \frac{1}{2}$ ,  $\xi_L$  converges to a clock process, and the fluctuation of the eigenvalue spacing converges to Gaussian. (ii)(critical case) for  $\alpha = \frac{1}{2}$ ,  $\xi_L$  converges to the limit of the circular  $\beta$ -ensemble.

Mathematics Subject Classification (2000): 60H25, 34L20

## 1 Introduction

### 1.1 Background

In this paper, we study the following Schrödinger operator

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on } L^2(\mathbf{R})$$

---

<sup>\*</sup>Kwanseigakuin University, Sanda 669-1337, Japan. e-mail : kotani@kwansei.ac.jp

<sup>†</sup>Department of Mathematics, Gakushuin University, 1-5-1, Mejiro, Toshima-ku, Tokyo, 171-8588, Japan. e-mail : fumihiko@math.gakushuin.ac.jp

where  $a \in C^\infty$  is real valued,  $a(-t) = a(t)$ , non-increasing for  $t \geq 0$ , and satisfies

$$C_1 t^{-\alpha} \leq a(t) \leq C_2 t^{-\alpha}$$

for some positive constants  $C_1, C_2$  and  $\alpha > 0$ .  $F$  is a real-valued, smooth, and non-constant function on a compact Riemannian manifold  $M$  such that

$$\langle F \rangle := \int_M F(x) dx = 0.$$

$\{X_t\}$  is a Brownian motion on  $M$ . Since the potential  $a(t)F(X_t)$  is  $-\frac{d^2}{dt^2}$ -compact, we have  $\sigma_{ess}(H) = [0, \infty)$ . Kotani-Ushiroya[3] proved that the spectrum of  $H$  in  $[0, \infty)$  is

- (1) for  $\alpha < \frac{1}{2}$  : pure point with exponentially decaying eigenfunctions,
- (2) for  $\alpha = \frac{1}{2}$  : pure point on  $[0, E_c]$  and purely singular continuous on  $[E_c, \infty)$  with some explicitly computable  $E_c$ ,
- (3) for  $\alpha > \frac{1}{2}$  : purely absolutely continuous.

In this paper we study the level statistics of this operator. For that purpose, let  $H_L := H|_{[0, L]}$  be the local Hamiltonian with Dirichlet boundary condition and let  $\{E_n(L)\}_{n=1}^\infty$  be its eigenvalues in the increasing order. Let  $n(L) \in \mathbf{N}$  be s.t.  $\{E_n(L)\}_{n \geq n(L)}$  coincides with the set of positive eigenvalues of  $H_L$ . We arbitrary take the reference energy  $E_0 > 0$  and consider the following point process

$$\xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}$$

in order to study the local fluctuation of eigenvalues near  $E_0$ . Our aim is to identify the limit of  $\xi_L$  as  $L \rightarrow \infty$ . Here we consider the scaling of  $\sqrt{E_n(L)}$ 's instead of  $E_n(L)$ 's. This corresponds to the unfolding with respect to the density of states.

This problem was first studied by Molchanov[6]. He proved that, when  $\alpha = 0$ ,  $\xi_L$  converges to the Poisson process. It was extended to the multidimensional Anderson model by [7]. Killip-Stoiciu [2] studied the CMV matrices whose matrix elements decay like  $n^{-\alpha}$ . They showed that,  $\xi_L$  converges to (i)  $\alpha > \frac{1}{2}$  : the clock process, (ii)  $\alpha = \frac{1}{2}$  : the limit of the circular  $\beta$ -ensemble, (iii)  $0 < \alpha < \frac{1}{2}$  : the Poisson process. Krichevski-Valko-Virag[5] studied the one-dimensional discrete Schrödinger operator with the random

potential decaying like  $n^{-1/2}$ , and proved that  $\xi_L$  converges to the Sine $_{\beta}$ -process.

The aim of our work is to do the analogue of that by Killip-Stoiciu[2] for the one-dimensional Schrödinger operator in the continuum.

In subsection 1.2 (resp. subsection 1.3), we state our results for ac-case :  $\alpha > \frac{1}{2}$  (resp. critical-case :  $\alpha = \frac{1}{2}$ ). We have not obtained results for pp-case :  $\alpha < \frac{1}{2}$ .

## 1.2 AC-case

**Definition 1.1** *Let  $\mu$  be a probability measure on  $[0, \pi)$ . We say that  $\xi$  is the clock process with spacing  $\pi$  with respect to  $\mu$  if and only if*

$$\mathbf{E}[e^{-\xi(f)}] = \int_0^\pi d\mu(\phi) \exp\left(-\sum_{n \in \mathbf{Z}} f(n\pi - \phi)\right)$$

where  $f \in C_c(\mathbf{R})$  and  $\xi(f) := \int_{\mathbf{R}} f d\xi$ .

We set

$$(x)_{\pi\mathbf{Z}} := x - [x]_{\pi\mathbf{Z}}, \quad [x]_{\pi\mathbf{Z}} := \max\{y \in \pi\mathbf{Z} \mid y \leq x\}.$$

We study the limit of  $\xi_L$  under the following assumption

**(A)**

(1)  $\alpha > \frac{1}{2}$ ,

(2) A sequence  $\{L_j\}_{j=1}^\infty$  satisfies  $\lim_{j \rightarrow \infty} L_j = \infty$  and

$$(\sqrt{E_0}L_j)_{\pi\mathbf{Z}} = \beta + o(1), \quad j \rightarrow \infty$$

for some  $\beta \in [0, \pi)$ .

The condition A(2) ensures that  $\xi_L$  converges to a point process. If  $a \equiv 0$  for instance, A(2) is indeed necessary.

**Theorem 1.1** *Assume (A). Then  $\xi_{L_j}$  converges in distribution to the clock process with spacing  $\pi$  with respect to a probability measure  $\mu_\beta$  on  $[0, \pi)$ .*

**Remark 1.1** Let  $x_t$  be the solution to the eigenvalue equation :  $H_L x_t = \kappa^2 x_t$  ( $\kappa > 0$ ). If we set

$$\begin{pmatrix} x_t \\ x_t'/\kappa \end{pmatrix} = \begin{pmatrix} r_t \sin \theta_t \\ r_t \cos \theta_t \end{pmatrix}, \quad \theta_t(\kappa) = \kappa t + \tilde{\theta}_t(\kappa),$$

then  $\tilde{\theta}_t(\kappa)$  has a limit as  $t$  goes to infinity[3] :  $\lim_{t \rightarrow \infty} \tilde{\theta}_t(\kappa) = \tilde{\theta}_\infty(\kappa)$ , a.s. ;  $\mu_\beta$  is the distribution of the random variable  $(\beta + \tilde{\theta}_\infty(\sqrt{E_0}))_{\pi \mathbf{Z}}$ . In some special cases, we can show that  $(\tilde{\theta}_\infty(\sqrt{E_0}))_{\pi \mathbf{Z}}$  is not uniformly distributed for large  $E_0$ , implying that  $\mu_\beta$  really depends on  $\beta$ .

**Remark 1.2** We can consider point processes with respect to two reference energies  $E_0, E'_0$  ( $E_0 \neq E'_0$ ) simultaneously : suppose a sequence  $\{L_j\}_{j=1}^\infty$  satisfies

$$(\sqrt{E_0} L_j)_{\pi \mathbf{Z}} = \beta + o(1), \quad (\sqrt{E'_0} L_j)_{\pi \mathbf{Z}} = \beta' + o(1), \quad j \rightarrow \infty$$

for some  $\beta, \beta' \in [0, \pi)$ . We set

$$\xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}, \quad \xi'_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E'_0})}.$$

Then the joint distribution of  $\xi_{L_j}, \xi'_{L_j}$  converges, for  $f, g \in C_c(\mathbf{R})$ ,

$$\begin{aligned} & \lim_{j \rightarrow \infty} \mathbf{E} [\exp(-\xi_{L_j}(f) - \xi'_{L_j}(g))] \\ &= \int_0^\pi d\mu(\phi, \phi') \exp \left( - \sum_{n \in \mathbf{Z}} (f(n\pi - \phi) + g(n\pi - \phi')) \right) \end{aligned}$$

where  $\mu(\phi, \phi')$  is the joint distribution of  $(\beta + \tilde{\theta}_\infty(\sqrt{E_0}))_{\pi \mathbf{Z}}$  and  $(\beta' + \tilde{\theta}_\infty(\sqrt{E'_0}))_{\pi \mathbf{Z}}$ . We are unable to identify  $\mu(\phi, \phi')$  but it may be possible that  $\phi$  and  $\phi'$  are correlated.

**Remark 1.3** Suppose we rearrange eigenvalues near the reference energy  $E_0$  so that

$$\dots < E'_{-2}(L) < E'_{-1}(L) < E_0 \leq E'_0(L) < E'_1(L) < E'_2(L) < \dots$$

Then an argument similar to the proof of Theorem 2.4 in [4] proves the following fact : for any  $n \in \mathbf{Z}$  we have

$$\lim_{L \rightarrow \infty} L(\sqrt{E'_{n+1}(L)} - \sqrt{E'_n(L)}) = \pi, \quad \text{a.s.} \quad (1.1)$$

which is called the strong clock behavior [1]. We note that the integrated density of states is equal to  $\sqrt{E}/\pi$ .

We next study the finer structure of the eigenvalue spacing, under the following assumption.

**(B)**

(1)  $\frac{1}{2} < \alpha < 1$ ,

(2) A sequence  $\{L_j\}_{j=1}^\infty$  satisfies  $\lim_{j \rightarrow \infty} L_j = \infty$  and

$$\sqrt{E_0}L_j = m_j\pi + \beta + \epsilon_j, \quad j \rightarrow \infty$$

for some  $\{m_j\}_{j=1}^\infty (\subset \mathbf{N})$ ,  $\beta \in [0, \pi)$  and  $\{\epsilon_j\}_{j=1}^\infty$  with  $\lim_{j \rightarrow \infty} \epsilon_j = 0$ .

(3)  $a(t) = t^{-\alpha}(1 + o(1))$ ,  $t \rightarrow \infty$ .

Roughly speaking,  $E_{m_j}(L_j)$  is the eigenvalue closest to  $E_0$ . In view of (1.1), we set

$$X_j(n) := \left\{ \left( \sqrt{E_{m_j+n+1}(L_j)} - \sqrt{E_{m_j+n}(L_j)} \right) L_j - \pi \right\} L_j^{\alpha-\frac{1}{2}}, \quad n \in \mathbf{Z}.$$

**Theorem 1.2** Assume (B). Then  $\{X_j(n)\}_{n \in \mathbf{Z}}$  converges in distribution to the Gaussian system with covariance

$$C(n, n') = \frac{C(E_0)}{8E_0} \operatorname{Re} \int_0^1 s^{-2\alpha} e^{2i(n-n')\pi s} 2(1 - \cos 2\pi s) ds, \quad n, n' \in \mathbf{Z},$$

where  $C(E) := \int_M \left| \nabla(L + 2i\sqrt{E})^{-1} F \right|^2 dx$  and  $L$  is the generator of  $(X_t)$ .

**Remark 1.4** Lemma 2.1 in [3] and Lemma 4.1 in section 4 imply that

$$\sqrt{E_{m_j}(L_j)} = \sqrt{E_0} - \frac{\beta + \tilde{\theta}_\infty(\sqrt{E_0})}{L_j} + Y_j$$

where  $Y_j = O(L_j^{-\alpha-\frac{1}{2}+\epsilon}) + O(\epsilon_j L_j^{-1})$ , a.s. for any  $\epsilon > 0$ . Furthermore by the definition of  $\{X_j(n)\}$  we have

$$\sqrt{E_{m_j+n}(L_j)} = \begin{cases} \sqrt{E_{m_j}(L_j)} + \frac{n\pi}{L_j} + \frac{1}{L_j^{\alpha+\frac{1}{2}}} \sum_{l=0}^{n-1} X_j(l) & (n \geq 1) \\ \sqrt{E_{m_j}(L_j)} + \frac{n\pi}{L_j} - \frac{1}{L_j^{\alpha+\frac{1}{2}}} \sum_{l=n}^{-1} X_j(l) & (n \leq -1) \end{cases}$$

and Theorem 1.2 thus describes the behavior of eigenvalues near  $E_{m_j}(L_j)$  in the second order.

**Remark 1.5** Suppose we consider two reference energies  $E_0, E'_0 (E_0 \neq E'_0)$  simultaneously and suppose a sequence  $\{L_j\}_{j=1}^\infty$  satisfies  $\lim_{j \rightarrow \infty} L_j = \infty$  and

$$\sqrt{E_0}L_j = m_j\pi + \beta + o(1), \quad \sqrt{E'_0}L_j = m'_j\pi + \beta' + o(1), \quad j \rightarrow \infty$$

for some  $m_j, m'_j \in \mathbf{N}$ , and  $\beta, \beta' \in [0, \pi)$ . Then  $\{X_j(n)\}_n$  and  $\{X'_j(n)\}_n$  converge jointly to the mutually independent Gaussian systems.

### 1.3 Critical Case

We set the following assumption.

$$(C) \quad a(t) = t^{-\frac{1}{2}}(1 + o(1)), \quad t \rightarrow \infty.$$

**Theorem 1.3** Assume (C). Then

$$\lim_{L \rightarrow \infty} \mathbf{E}[e^{-\xi_L(f)}] = \mathbf{E} \left[ \int_0^{2\pi} \frac{d\theta}{2\pi} \exp \left( - \sum_{n \in \mathbf{Z}} f(\Psi_1^{-1}(2n\pi + \theta)) \right) \right]$$

where  $\{\Psi_t(\cdot)\}_{t \geq 0}$  is the strictly-increasing function valued process such that for any  $c_1, \dots, c_m \in \mathbf{R}$ ,  $\{\Psi_t(c_j)\}_{j=1}^m$  is the unique solution of the following SDE :

$$\begin{aligned} d\Psi_t(c_j) &= 2c_j dt + D \operatorname{Re} \left\{ (e^{i\Psi_t(c_j)} - 1) \frac{dZ_t}{\sqrt{t}} \right\} \\ \Psi_0(c_j) &= 0, \quad j = 1, 2, \dots, m \end{aligned}$$

where  $C(E_0) := \int_M |\nabla(L + 2i\sqrt{E_0})^{-1}F|^2 dx$ ,  $D := \sqrt{\frac{C(E_0)}{2E_0}}$  and  $Z_t$  is a complex Brownian motion.

**Definition 1.2** For  $\beta > 0$ , the circular  $\beta$ -ensemble with  $n$ -points is given by

$$\mathbf{E}_n^\beta[G] := \frac{1}{Z_{n,\beta}} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{d\theta_n}{2\pi} G(\theta_1, \dots, \theta_n) |\Delta(e^{i\theta_1}, \dots, e^{i\theta_n})|^\beta$$

where  $Z_{n,\beta}$  is the normalization constant,  $G \in C(\mathbf{T}^n)$  is bounded and  $\Delta$  is the Vandermonde determinant. The limit  $\xi_\beta$  of the circular  $\beta$ -ensemble is defined

$$\mathbf{E}[e^{-\xi_\beta(f)}] = \lim_{n \rightarrow \infty} \mathbf{E}_n^\beta \left[ \exp \left( - \sum_{j=1}^n f(n\theta_j) \right) \right], \quad f \in C_c^+(\mathbf{R})$$

whose existence and characterization is given by [2]. The result in [2] together with Theorem 1.3 imply the limit of  $\xi_L$  coincides with that of the circular  $\beta$ -ensemble modulo a scaling.

**Corollary 1.4** *Assume (C). Writing  $\xi_\beta = \sum_n \delta_{\lambda_n}$ , let  $\xi'_\beta := \sum_n \delta_{\lambda_n/2}$ . Then  $\xi_L \xrightarrow{d} \xi'_\beta$  with  $\beta = \beta(E_0) := \frac{8E_0}{C(E_0)}$ .*

**Remark 1.6** *The corresponding  $\beta = \beta(E_0) = \frac{8E_0}{C(E_0)}$  depends on the reference energy  $E_0$ , so that the spacing distribution may change if we look at the different region in the spectrum. To see how  $\beta$  changes, we recall some results in [3]. Let  $\sigma_F(\lambda)$  be the spectral measure of the generator  $L$  of  $\{X_t\}$  with respect to  $F$ . Then*

$$\gamma(E) := -\frac{1}{4E} \int_{-\infty}^0 \frac{\lambda}{\lambda^2 + 4E} d\sigma_F(\lambda), \quad E > 0$$

*is the Lyapunov exponent in the sense that any generalized eigenfunction  $\psi_E$  of  $H$  satisfies*

$$\lim_{|t| \rightarrow \infty} (\log t)^{-1} \log \{ \psi_E(t)^2 + \psi'_E(t)^2 \}^{1/2} = -\gamma(E), \quad a.s..$$

*Moreover  $E < E_c$  (resp.  $E > E_c$ ) if and only if  $\gamma(E) > \frac{1}{2}$  (resp.  $\gamma(E) < \frac{1}{2}$ ) and  $\gamma(E_c) = \frac{1}{2}$ . Since  $C(E) = 8E \cdot \gamma(E)$ , we have  $\beta(E) = \frac{1}{\gamma(E)}$ . It then follows that  $E < E_c$  (resp.  $E > E_c$ ) if and only if  $\beta(E) < 2$  (resp.  $\beta(E) > 2$ ) and  $\beta(E_c) = 2$ . This is consistent with our general belief that in the point spectrum (resp. in the continuous spectrum) the level repulsion is weak (resp. strong). We also note that if  $\beta = 2$ , the circular  $\beta$ -ensemble with  $n$ -points coincides with the eigenvalue distribution of the unitary ensemble with the Haar measure on  $U(n)$ .*

**Remark 1.7** *If we consider two reference energies  $E_0, E'_0 (E_0 \neq E'_0)$ , then the corresponding point process  $\xi_L, \xi'_L$  converges jointly to the independent  $\xi_\beta, \xi'_{\beta'}$ .*

In later sections, we prove theorems mentioned above based on the argument in [2, 3, 4] : The main ingredient of the proof is to study the limiting behavior of the relative Prüfer phase. In section 2 we prepare some notations and basic facts. In sections 3, 4, we consider the ac-case and prove Theorems 1.1, 1.2. In sections 6-9, we consider the critical case and prove Theorem 1.3 which is outlined in section 5. In what follows,  $C$  denotes general positive constant which is subject to change from line to line in each argument.

## 2 Preliminaries

Let  $x_t$  be the solution to the equation  $H_L x_t = \kappa^2 x_t$  ( $\kappa > 0$ ) which we set in the following form

$$\begin{pmatrix} x_t \\ x'_t/\kappa \end{pmatrix} = r_t \begin{pmatrix} \sin \theta_t \\ \cos \theta_t \end{pmatrix}, \quad \theta_0 = 0.$$

We define  $\tilde{\theta}_t(\kappa)$  by

$$\theta_t(\kappa) = \kappa t + \tilde{\theta}_t(\kappa).$$

Then it follows that

$$r_t(\kappa) = \exp \left( \frac{1}{2\kappa} \operatorname{Im} \int_0^t a(s) F(X_s) e^{2i\theta_s(\kappa)} ds \right) \quad (2.1)$$

$$\tilde{\theta}_t(\kappa) = \frac{1}{2\kappa} \int_0^t \operatorname{Re}(e^{2i\theta_s(\kappa)} - 1) a(s) F(X_s) ds \quad (2.2)$$

$$\frac{\partial \theta_t(\kappa)}{\partial \kappa} = \int_0^t \frac{r_s^2}{r_t^2} ds + \frac{1}{2\kappa^2} \int_0^t \frac{r_s^2}{r_t^2} a(s) F(X_s) (1 - \operatorname{Re} e^{2i\theta_s(\kappa)}) ds. \quad (2.3)$$

Since  $\frac{\partial \theta_t(\kappa)}{\partial \kappa} > 0$ ,  $\theta_t(\kappa)$  is increasing as a function of  $\kappa$ . Here and henceforth, for simplicity, we say  $f$  is increasing if and only if  $x < y$  implies  $f(x) < f(y)$ . Set

$$\begin{aligned} \theta_L(\sqrt{E_0}) &= m(E_0, L)\pi + \phi(E_0, L) \\ m(E_0, L)\pi &:= [\theta_L(\sqrt{E_0})]_{\pi\mathbf{Z}}, \quad \phi(E_0, L) := (\theta_L(\sqrt{E_0}))_{\pi\mathbf{Z}} \in [0, \pi). \end{aligned} \quad (2.4)$$



Moreover we define “the relative Prüfer phase”

$$\Psi_L(x) = \theta_L(\sqrt{E_0} + \frac{x}{L}) - \theta_L(\sqrt{E_0}).$$

As is done in [2] we use the following representation of the Laplace transform of  $\xi_L$  in terms of  $\Psi_L$ .

**Lemma 2.1** *For  $f \in C_c^+(\mathbf{R})$  we have*

$$\mathbf{E}[e^{-\xi_L(f)}] = \mathbf{E} \left[ \exp \left( - \sum_{n=n(L)-m(E_0, L)}^{\infty} f \left( \Psi_L^{-1}(n\pi - \phi(E_0, L)) \right) \right) \right].$$

*Proof.* Let

$$x_n(L) := L(\sqrt{E_n(L)} - \sqrt{E_0}), \quad n \geq n(L)$$

be the atoms of  $\xi_L$ . Since  $\theta_L(\sqrt{E_n(L)}) = \theta_L(\sqrt{E_0} + \frac{x_n}{L}) = n\pi$  we have

$$\Psi_L(x_n) = (n - m(E_0, L))\pi - \phi(E_0, L).$$

Here we note that  $\Psi_L(x)$  is continuous and increasing, and thus has the inverse  $\Psi_L^{-1}$ .  $\square$

### 3 Convergence to a clock process

In what follows we set

$$\kappa := \sqrt{E_0}$$

for simplicity.

#### 3.1 The behavior of $\Psi_L$

**Proposition 3.1** *If  $\alpha > \frac{1}{2}$ , following fact holds for a.s. :*

$$\lim_{L \rightarrow \infty} \Psi_L(x) = x$$

*pointwise and this holds compact uniformly with respect to  $\kappa$ .*

*Proof.* By (2.2) we have

$$\begin{aligned}\Psi_L(x) &= x + \left( \frac{1}{2(\kappa + \frac{x}{L})} - \frac{1}{2\kappa} \right) \int_0^L \operatorname{Re} \left( e^{2i\theta_s(\kappa + \frac{x}{L})} - 1 \right) a(s) F(X_s) ds \\ &\quad + \frac{1}{2\kappa} \operatorname{Re} \int_0^L a(s) F(X_s) \left( e^{2i\theta_s(\kappa + \frac{x}{L})} - e^{2i\theta_s(\kappa)} \right) ds \\ &=: x + I + II.\end{aligned}$$

Since  $a(s) = O(x^{-\alpha})$ ,  $|x| \rightarrow \infty$ ,

$$I = -\frac{1}{2} \cdot \frac{\frac{x}{L}}{\kappa(\kappa + \frac{x}{L})} \int_0^L \operatorname{Re} \left( e^{2i\theta_s(\kappa + \frac{x}{L})} - 1 \right) a(s) F(X_s) ds = O(L^{-\alpha}).$$

We next set

$$A_t(\kappa, \beta) := \int_0^t a(s) F(X_s) e^{i\beta\theta_s(\kappa)} ds.$$

Take  $\delta > 0$  such that

$$\int_0^\infty a(s)^2 s^\delta ds < \infty,$$

then by [3] Lemma 2.2  $A_t(\kappa, \beta)$  has a limit as  $t \rightarrow \infty$  :  $\lim_{t \rightarrow \infty} A_t(\kappa, \beta) = A_\infty(\kappa, \beta)$  compact uniformly w.r.t.  $\kappa$ . Moreover, for any compact set  $K \subset (0, \infty)$  and for any  $\epsilon < \frac{\delta}{2}$ ,  $\beta \in \mathbf{R}$  we have

$$\sup_{t \geq 0, \kappa, \kappa_1 \in K} \frac{|A_t(\kappa, \beta) - A_t(\kappa_1, \beta)|}{|\kappa - \kappa_1|^\epsilon} < \infty, \quad a.s..$$

Due to this fact we have, for fixed  $x$ ,

$$II = \frac{1}{2\kappa} \operatorname{Re} \left( A_L(\kappa + \frac{x}{L}, 2) - A_L(\kappa, 2) \right) = O(L^{-\epsilon}), \quad a.s..$$

It then suffices to use Lemma 3.2 stated below.  $\square$

**Lemma 3.2** *Let  $\Phi_1, \Phi_2, \dots$ , and  $\Phi$  be the non-decreasing functions s.t.  $\Phi_n(x) \rightarrow \Phi(x)$  for  $x \in \mathbf{Q}$ . Then  $\Phi_n(x) \rightarrow \Phi(x)$  for any continuity point  $x \in \mathbf{R}$  of  $\Phi$ .*

## 3.2 Proof of Theorem 1.1

We sometimes use the following elementary lemma.

**Lemma 3.3** *Let  $\Psi_n, n = 1, 2, \dots$ , and  $\Psi$  are continuous and increasing functions such that  $\lim_{n \rightarrow \infty} \Psi_n(x) = \Psi(x)$  pointwise. If  $y_n \in \text{Ran} \Psi_n$ ,  $y \in \text{Ran} \Psi$  and  $y_n \rightarrow y$ , then it holds that*

$$\Psi_n^{-1}(y_n) \xrightarrow{n \rightarrow \infty} \Psi^{-1}(y).$$

*Proof of Theorem 1.1*

$\tilde{\theta}_t(\kappa) \xrightarrow{t \rightarrow \infty} \tilde{\theta}_\infty(\kappa) + o(1)$  by [3] Proposition 2.1 and  $\kappa L_j = m_j \pi + \beta + o(1)$  for some  $m_j \in \mathbf{N}$  by Assumption (A). Thus we have  $\theta_{L_j}(\kappa) = m_j \pi + \beta + \tilde{\theta}_\infty(\kappa) + o(1)$  and hence

$$\lim_{\kappa \rightarrow \infty} \phi(\kappa^2, L_j) = \phi_\beta := \left( \tilde{\theta}_\infty(\kappa) + \beta \right)_{\pi \mathbf{Z}}, \quad a.s.. \quad (3.1)$$

By (3.1) and Proposition 3.1, the assumption for Lemma 3.3 is satisfied. Thus we use Lemma 2.1 and the fact that  $\lim_{L \rightarrow \infty} (n(L) - m(\kappa^2, L)) = -\infty$  to conclude that

$$\begin{aligned} \mathbf{E}[e^{-\xi_{L_j}(f)}] &= \mathbf{E} \left[ \exp \left( - \sum_{n=n(L_j)-m(\kappa^2, L_j)}^{\infty} f(\Psi_{L_j}^{-1}(n\pi - \phi(\kappa^2, L_j))) \right) \right] \\ &\xrightarrow{j \rightarrow \infty} \mathbf{E} \left[ \exp \left( - \sum_{n \in \mathbf{Z}} f(n\pi - \phi_\beta) \right) \right] \\ &= \int d\mu_\beta(\phi) \exp \left( - \sum_{n \in \mathbf{Z}} f(n\pi - \phi) \right) \end{aligned}$$

for  $f \in C_c^+(\mathbf{R})$  where  $\mu_\beta$  is the distribution of  $\phi_\beta$ .  $\square$

## 4 Second Limit Theorem

### 4.1 Behavior of eigenvalues near $E_0$

**Lemma 4.1** *Assume (B) and let  $n \in \mathbf{Z}$ . Then for  $j \rightarrow \infty$  we have*

$$\begin{aligned} (1) \quad & \sqrt{E_{m_j+n}(L_j)} = \kappa + o(1) \\ (2) \quad & \sqrt{E_{m_j+n}(L_j)} = \kappa + \frac{n\pi - \beta - \tilde{\theta}_\infty(\kappa)}{L_j} + o(L_j^{-1}). \end{aligned}$$

*Proof.* (1) This easily follows from the two equations given below.

$$\begin{aligned}(m_j + n)\pi &= \theta_{L_j}(\sqrt{E_{m_j+n}(L_j)}) \\ &= \sqrt{E_{m_j+n}(L_j)}L_j + \tilde{\theta}_{L_j}(\sqrt{E_{m_j+n}(L_j)})\end{aligned}\quad (4.1)$$

$$(m_j + n)\pi = \kappa L_j - \beta + n\pi + o(1). \quad (4.2)$$

(2) We substitute Lemma 4.1(1) into the last term in the RHS of (4.1). Since the convergence  $\tilde{\theta}_t(\kappa) \rightarrow \tilde{\theta}_\infty(\kappa)$  holds compact uniformly with respect to  $\kappa$  [3] we have

$$(m_j + n)\pi = \sqrt{E_{m_j+n}(L_j)}L_j + \tilde{\theta}_\infty(\kappa) + o(1). \quad (4.3)$$

Lemma 4.1(2) follows from (4.2) and (4.3).  $\square$

By taking the difference between

$$\begin{aligned}(m_j + n + 1)\pi &= \sqrt{E_{m_j+n+1}(L)} \cdot L + \tilde{\theta}_L(\sqrt{E_{m_j+n+1}(L)}) \\ (m_j + n)\pi &= \sqrt{E_{m_j+n}(L)} \cdot L + \tilde{\theta}_L(\sqrt{E_{m_j+n}(L)})\end{aligned}$$

we see that

$$X_j(n) = -L_j^{\alpha-\frac{1}{2}} \left( \tilde{\theta}_{L_j}(\sqrt{E_{m_j+n+1}(L_j)}) - \tilde{\theta}_{L_j}(\sqrt{E_{m_j+n}(L_j)}) \right).$$

By Lemma 4.1(2)

$$\begin{aligned}\sqrt{E_{m_j+n+1}(L_j)} &= \kappa + \frac{c_1}{L_j}, \quad \sqrt{E_{m_j+n}(L_j)} = \kappa + \frac{c_2}{L_j} \\ c_1 &= (n+1)\pi - \beta - \tilde{\theta}_\infty(\kappa) + o(1) \\ c_2 &= n\pi - \beta - \tilde{\theta}_\infty(\kappa) + o(1), \quad k \rightarrow \infty.\end{aligned}$$

We set

$$\begin{aligned}\Theta_t^{(n)}(c_1, c_2) &:= \left( \tilde{\theta}_{nt}(\kappa + \frac{c_1}{n}) - \tilde{\theta}_{nt}(\kappa + \frac{c_2}{n}) \right) n^{\alpha-\frac{1}{2}} \\ l_t((c_1, c_2), (c'_1, c'_2)) &:= \frac{C(\kappa^2)}{8\kappa^2} \int_0^t s^{-2\alpha} \operatorname{Re} \left( e^{2ic_1s} - e^{2ic_2s} \right) \overline{(e^{2ic'_1s} - e^{2ic'_2s})} ds.\end{aligned}$$

When  $c_1, c_2$  are constant, the following fact is proved in [4] Lemma 3.1.

**Proposition 4.2**  $\{\Theta_t^{(n)}(c_1, c_2)\}_{t \geq 0, c_1, c_2 \in \mathbf{R}} \xrightarrow{d} \{Z(t, c_1, c_2)\}_{t \geq 0, c_1, c_2 \in \mathbf{R}}$  as  $n \rightarrow \infty$  where  $\{Z(t, c_1, c_2)\}_{t \geq 0, c_1, c_2 \in \mathbf{R}}$  is the Gaussian system with covariance  $l_{t \wedge t'}((c_1, c_2), (c'_1, c'_2))$ .

## 4.2 Independence of the limits

To finish the proof of Theorem 1.2, it is then sufficient to prove that  $(\tilde{\theta}_{nt}(\kappa), \{\Theta_t^{(n)}(c_1, c_2)\}_{c_1, c_2})$  converges jointly to the independent ones. Let  $0 < \kappa_1 < \kappa_2$  and  $I := [\kappa_1, \kappa_2]$ . In the following lemma, we regard  $\tilde{\theta}_t, \tilde{\theta}_\infty$  are  $C(I)$ -valued random elements.

**Lemma 4.3** *For  $t > 0$  fixed, we have*

$$(\tilde{\theta}_{nt}, \{\Theta_t^{(n)}(c_1, c_2)\}_{c_1, c_2}) \xrightarrow{d} (\tilde{\theta}_\infty, \{Z(t, c_1, c_2)\}_{c_1, c_2})$$

as  $n \rightarrow \infty$  where  $\tilde{\theta}_\infty$  and  $\{Z(t, c_1, c_2)\}_{c_1, c_2}$  are independent.

*Proof.* Let  $A(\subset C(I))$  be a  $\tilde{\theta}_\infty$ -continuity set (i.e.,  $\mathbf{P}(\tilde{\theta}_\infty \in \partial A) = 0$ ) and set  $A_\epsilon := \{f \in C(I) \mid d(f, A) < \epsilon\}$ . Since  $\tilde{\theta}_t(\kappa) \xrightarrow{a.s.} \tilde{\theta}_\infty(\kappa)$  compact uniformly in  $\kappa$ , for any  $\epsilon > 0$

$$\mathbf{P}(\tilde{\theta}_{nt} \in A, \tilde{\theta}_T \notin A_\epsilon) = o(1)$$

for sufficiently large  $T, n$ .

Here we recall eq.(3.3) in [4].

$$\begin{aligned} \Theta_t^{(n)}(c_1, c_2) &= T_t^{(n)}(c_1, c_2) + O(n^{\frac{1}{2}-\alpha}) \\ \text{where } T_t^{(n)}(c_1, c_2) &:= n^{\alpha-\frac{1}{2}} \operatorname{Re} \left( S_t^{(n)}(\kappa_1) - S_t^{(n)}(\kappa_2) \right) \\ S_t^{(n)}(\kappa) &:= \frac{1}{2\kappa} \int_0^{nt} a(s) e^{2i\tilde{\theta}_s(\kappa)} dM_s(\kappa) \end{aligned}$$

$M_s(\kappa)$  is the complex martingale defined in subsection 6.2.

Let  $m \in \mathbf{N}$ . For  $\mathbf{c}_1 = (c_1^{(1)}, \dots, c_1^{(m)})$ ,  $\mathbf{c}_2 = (c_2^{(1)}, \dots, c_2^{(m)})$ , we use the following convention :  $\Theta_t^{(n)}(\mathbf{c}_1, \mathbf{c}_2) = (\Theta_t^{(n)}(c_1^{(1)}, c_2^{(1)}), \dots, \Theta_t^{(n)}(c_1^{(m)}, c_2^{(m)}))$  and similarly for  $T_t^{(n)}(\mathbf{c}_1, \mathbf{c}_2)$  and  $Z(t, \mathbf{c}_1, \mathbf{c}_2)$ . Let  $B \in \mathcal{B}(\mathbf{R}^m)$  be a  $Z(t, \mathbf{c}_1, \mathbf{c}_2)$ -continuity set and let  $B_\epsilon := \{x \in \mathbf{R}^m \mid d(x, B) < \epsilon\}$ . Writing  $\Theta_t^{(n)} = \Theta_t^{(n)}(\mathbf{c}_1, \mathbf{c}_2)$ ,  $T_t^{(n)} = T_t^{(n)}(\mathbf{c}_1, \mathbf{c}_2)$  we have, for sufficiently large  $n$ ,

$$\begin{aligned} \mathbf{P}(\tilde{\theta}_{nt} \in A, \Theta_t^{(n)} \in B) &\leq \mathbf{P}(\tilde{\theta}_T \in A_\epsilon, T_t^{(n)} \in B_\epsilon) + o(1) \\ &= \mathbf{P}(\tilde{\theta}_T \in A_\epsilon, T_t^{(n)} - T_{T/n}^{(n)} + T_{T/n}^{(n)} \in B_\epsilon) + o(1) \\ &= \mathbf{P}(\tilde{\theta}_T \in A_\epsilon, T_t^{(n)} - T_{T/n}^{(n)} \in B_{2\epsilon}) + o(1). \end{aligned}$$

Here we used  $T_{T/n}^{(n)} \xrightarrow{P} 0$ . By the Markov property

$$= \mathbf{E} \left[ 1_{\{\tilde{\theta}_T \in A_\epsilon\}} \mathbf{E}_{X_T} \left[ 1_{\{\tilde{T}_{t-T/n}^{(n)} \in B_{2\epsilon}\}} \right] \right] + o(1)$$

where  $\tilde{T}_t^{(n)}$  is the suitable “time-shift” of  $T_t^{(n)}$ . Because  $\tilde{T}_t^{(n)}$  converges in distribution to the Gaussian  $Z(t, \mathbf{c}_1, \mathbf{c}_2)$  as  $n \rightarrow \infty$  being irrespective of  $X_T$ ,

$$\begin{aligned} &= \mathbf{P} \left( \tilde{\theta}_T \in A_\epsilon \right) \mathbf{P} \left( Z(t, \mathbf{c}_1, \mathbf{c}_2) \in B_{2\epsilon} \right) + o(1) \\ &\leq \mathbf{P} \left( \tilde{\theta}_\infty \in A_{2\epsilon} \right) \mathbf{P} \left( Z(t, \mathbf{c}_1, \mathbf{c}_2) \in B_{2\epsilon} \right) + o(1) \end{aligned}$$

Since  $A$  is a  $\tilde{\theta}_\infty$ -continuity set and  $B \in \mathcal{B}(\mathbf{R}^m)$  is a  $Z(t, \mathbf{c}_1, \mathbf{c}_2)$ -continuity set,

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left( \tilde{\theta}_{nt} \in A, \Theta_t^{(n)} \in B \right) \leq \mathbf{P}(\tilde{\theta}_\infty \in A) \mathbf{P}(Z(t, \mathbf{c}_1, \mathbf{c}_2) \in B).$$

The opposite inequality can be proved similarly.  $\square$

## 5 SC-case Outline of proof of Theorem 1.3

In this section we overview the proof of Theorem 1.3. First of all, set

$$\begin{aligned} (x)_{2\pi\mathbf{Z}} &:= x - [x]_{2\pi\mathbf{Z}}, \quad [x]_{2\pi\mathbf{Z}} := \max\{y \in 2\pi\mathbf{Z} \mid y \leq x\} \\ 2m(\kappa^2, L)\pi &:= [2\theta_L(\kappa)]_{2\pi\mathbf{Z}} \\ \phi(\kappa^2, L) &:= (2\theta_L(\kappa))_{2\pi\mathbf{Z}} \in [0, 2\pi) \end{aligned}$$

We also set “the relative Prüfer phase” by

$$\Phi_L(x) := 2\theta_L\left(\kappa + \frac{x}{L}\right) - 2\theta_L(\kappa).$$

Then we have a variant of Lemma 2.1.

**Lemma 5.1** *For  $f \in C_c^+(\mathbf{R})$*

$$\mathbf{E}[e^{-\xi_L(f)}] = \mathbf{E} \left[ \exp \left( - \sum_{n=n(L)-m(\kappa^2, L)}^{\infty} f\left(\Phi_L^{-1}(2n\pi - \phi(\kappa^2, L))\right) \right) \right].$$

So our task is to study the limit of the joint distribution of  $(\Phi_L, \phi(\kappa^2, L))$  as  $L \rightarrow \infty$ . Following [2] we consider

$$\Psi_t^{(n)}(x) := 2\theta_{nt}(\kappa + \frac{x}{n}) - 2\theta_{nt}(\kappa),$$

regard it as an increasing function-valued process, and find a process  $\Psi_t(x)$  such that for any fixed  $c_1, \dots, c_m \in \mathbf{R}$   $\{\Psi_t^{(n)}(c_j)\}_{j=1}^m \xrightarrow{d} \{\Psi_t(c_j)\}_{j=1}^m$  (Theorem 6.10).  $\Psi_t$  is characterized as the unique solution to the following SDE.

$$\begin{aligned} d\Psi_t(c) &= 2cdt + \frac{2}{\sqrt{\beta t}} \text{Re} [(e^{i\Psi_t(c)} - 1)dZ_t] \\ \Psi_0(c) &= 0, \quad \beta = \frac{8\kappa^2}{C(\kappa^2)}. \end{aligned} \tag{5.1}$$

Moreover,  $\Psi_t(c)$  is continuous and increasing with respect to  $c$  (Lemma 6.11). On the other hand we have  $(\{\Psi_1^{(n)}(c_j)\}_{j=1}^m, \phi(\kappa^2, n)) \xrightarrow{d} (\{\Psi_1(c_j)\}_{j=1}^m, \phi_1)$  jointly, where  $\phi_1$  is uniformly distributed on  $[0, 2\pi)$  and independent of  $\Psi_1$  (Proposition 9.1). Moreover  $\Psi^{(n)}$  converges to  $\Psi$  also as a sequence of increasing function-valued process (Lemma 9.3), so that we can find a coupling such that for a.s.  $((\Psi_1^{(n)})^{-1}(x), \phi(\kappa^2, n)) \rightarrow (\Psi_1^{-1}(x), \phi_1)$  for any  $x \in \mathbf{R}$  (Proposition 9.2). Therefore

$$\lim_{L \rightarrow \infty} \mathbf{E}[e^{-\xi_L(f)}] = \mathbf{E} \left[ \int_0^{2\pi} \frac{d\phi_1}{2\pi} \exp \left( - \sum_{n \in \mathbf{Z}} f(\Psi_1^{-1}(2n\pi - \phi_1)) \right) \right]$$

which coincides with what is derived in [2] (except that the drift term  $cdt$  in [2] is replaced by  $2cdt$  that is why we need to consider a scaling :  $\xi'_\beta = \sum_n \delta_{\lambda_n/2}$ ) thereby identifying the limit of  $\xi_L$  with that of the circular  $\beta$ -ensemble.

## 6 Convergence of $\Psi$

### 6.1 Preliminaries

For  $f \in C^\infty(M)$  let  $R_\beta f := (L + i\beta)^{-1} f$  ( $\beta > 0$ ),  $Rf := L^{-1}(f - \langle f \rangle)$ . Then by Ito's formula,

$$\begin{aligned} \int_0^t e^{i\beta s} f(X_s) ds &= [e^{i\beta s} (R_\beta f)(X_s)]_0^t + \int_0^t e^{i\beta s} dM_s(f, \beta) \\ \int_0^t f(X_s) ds &= \langle f \rangle t + [(Rf)(X_s)]_0^t + M_t(f, 0) \end{aligned}$$

where  $M_s(f, \beta), M_s(f, 0)$  are the complex martingales whose variational process satisfy

$$\begin{aligned}\langle M(f, \beta), M(f, \beta) \rangle_t &= \int_0^t [R_\beta f, R_\beta f](X_s) ds, \\ \langle M(f, \beta), \overline{M(f, \beta)} \rangle_t &= \int_0^t [R_\beta f, \overline{R_\beta f}](X_s) ds \\ \langle M(f, 0), M(f, 0) \rangle_t &= \int_0^t [Rf, Rf](X_s) ds, \\ \langle M(f, 0), \overline{M(f, 0)} \rangle_t &= \int_0^t [Rf, \overline{Rf}](X_s) ds\end{aligned}$$

where

$$\begin{aligned}[f_1, f_2](x) &:= L(f_1 f_2)(x) - (Lf_1)(x)f_2(x) - f_1(x)(Lf_2)(x) \\ &= (\nabla f_1, \nabla f_2)(x).\end{aligned}$$

Then the integration by parts gives us the following formulas to be used frequently.

**Lemma 6.1**

$$\begin{aligned}(1) \quad & \int_0^t b(s) e^{i\beta s} e^{i\gamma \tilde{\theta}_s} f(X_s) ds \\ &= \left[ b(s) e^{i\gamma \tilde{\theta}_s} e^{i\beta s} (R_\beta f)(X_s) \right]_0^t - \int_0^t b'(s) e^{i\gamma \tilde{\theta}_s} e^{i\beta s} (R_\beta f)(X_s) ds \\ &\quad - \frac{i\gamma}{2\kappa} \int_0^t b(s) a(s) \operatorname{Re}(e^{2i\theta_s} - 1) e^{i\gamma \tilde{\theta}_s} e^{i\beta s} F(X_s) (R_\beta f)(X_s) ds \\ &\quad + \int_0^t b(s) e^{i\beta s} e^{i\gamma \tilde{\theta}_s} dM_s(f, \beta).\end{aligned}$$

$$\begin{aligned}(2) \quad & \int_0^t b(s) e^{i\gamma \tilde{\theta}_s} f(X_s) ds \\ &= \langle f \rangle \int_0^t b(s) e^{i\gamma \tilde{\theta}_s} ds \\ &\quad + \left[ b(s) e^{i\gamma \tilde{\theta}_s} (Rf)(X_s) \right]_0^t - \int_0^t b'(s) e^{i\gamma \tilde{\theta}_s} (Rf)(X_s) ds\end{aligned}$$



$$\begin{aligned}
& -\frac{i\gamma}{2\kappa} \int_0^t a(s)b(s)Re(e^{2i\theta_s} - 1)e^{i\gamma\bar{\theta}_s}F(X_s)(Rf)(X_s)ds \\
& + \int_0^t b(s)e^{i\gamma\bar{\theta}_s}dM_s(f, 0).
\end{aligned}$$

We will also use following notation for simplicity.

$$\begin{aligned}
g_\kappa &:= (L + 2i\kappa)^{-1}F, \quad g := L^{-1}(F - \langle F \rangle), \\
M_s(\kappa) &:= M_s(F, 2\kappa), \quad M_s := M_s(F, 0).
\end{aligned}$$

## 6.2 A priori estimates

In this section we derive a priori estimate for the following quantity

$$\Psi_t^{(n)}(c) := 2\theta_{nt}(\kappa + \frac{c}{n}) - 2\theta_{nt}(\kappa).$$

By (2.2) we have

$$\begin{aligned}
\Psi_t^{(n)}(c) &= 2ct + \frac{1}{(\kappa + \frac{c}{n})} \int_0^{nt} Re(e^{2i\theta_s(\kappa + \frac{c}{n})} - 1)a(s)F(X_s)ds \\
&\quad - \frac{1}{\kappa} \int_0^{nt} Re(e^{2i\theta_s(\kappa)} - 1)a(s)F(X_s)ds \\
&= 2ct - \frac{\frac{c}{n}}{\kappa(\kappa + \frac{c}{n})} \int_0^{nt} Re(e^{2i\theta_s(\kappa + \frac{c}{n})} - 1)a(s)F(X_s)ds \\
&\quad + \frac{1}{\kappa} Re \int_0^{nt} (e^{2i\theta_s(\kappa + \frac{c}{n})} - e^{2i\theta_s(\kappa)})a(s)F(X_s)ds \quad (6.1)
\end{aligned}$$

and the third term in the RHS will be dominant.

**Lemma 6.2** *Suppose*

$$\int_0^\infty a(s)^4 ds < \infty$$

*we then have*

(1)

$$\begin{aligned}
& \int_0^t a(s)e^{2i\theta_s(\kappa)}F(X_s)ds \\
&= -\frac{i}{2\kappa} \int_0^t a(s)^2 g_\kappa(X_s)F(X_s)ds + Y_t(\kappa) + \delta_t(\kappa)
\end{aligned}$$

where

$$\begin{aligned}
Y_t(\kappa) &:= \int_0^t a(s) e^{2i\theta_s(\kappa)} dM_s(\kappa) \\
\delta_t(\kappa) &:= [a(s) e^{2i\theta_s(\kappa)} g_\kappa(X_s)]_0^t - \int_0^t a'(s) e^{2i\theta_s(\kappa)} g_\kappa(X_s) ds \\
&\quad - \frac{i}{\kappa} \int_0^t a(s)^2 e^{2i\theta_s(\kappa)} \left( \frac{e^{2i\theta_s(\kappa)}}{2} - 1 \right) g_\kappa(X_s) F(X_s) ds.
\end{aligned}$$

(2) For a.s.,  $\delta_t(\kappa)$  has the limit as  $t \rightarrow \infty$   $F \lim_{t \rightarrow \infty} \delta_t(\kappa) = \delta_\infty(\kappa)$ , a.s.

(3) For any  $0 < T < \infty$ , we have

$$\mathbf{E} \left[ \max_{0 \leq t \leq T} \left| \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right|^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* (1) By Lemma 6.1(1)

$$\begin{aligned}
&\int_0^t a(s) e^{2i\theta_s(\kappa)} F(X_s) ds \\
&= [a(s) e^{2i\theta_s(\kappa)} g_\kappa(X_s)]_0^t - \int_0^t a'(s) e^{2i\theta_s(\kappa)} g_\kappa(X_s) ds \\
&\quad - \frac{2i}{2\kappa} \int_0^t a(s)^2 \operatorname{Re}(e^{2i\theta_s(\kappa)} - 1) e^{2i\theta_s(\kappa)} g_\kappa(X_s) F(X_s) ds \\
&\quad + \int_0^t a(s) e^{2i\theta_s(\kappa)} dM_s(\kappa)
\end{aligned}$$

which we decompose into “non-oscillating” term + martingale-term + remainder.

$$= -\frac{i}{2\kappa} \int_0^t a(s)^2 g_\kappa(X_s) ds + Y_t(\kappa) + \delta_t(\kappa)$$

where the remainder term  $\delta_t(\kappa)$  is further decomposed for later use

$$\delta_t(\kappa) = \delta_t^{(1)}(\kappa) + \delta_t^{(2)}(\kappa) \tag{6.2}$$

$$\delta_t^{(1)}(\kappa) := [a(s) e^{2i\theta_s(\kappa)} g_\kappa(X_s)]_0^t - \int_0^t a'(s) e^{2i\theta_s(\kappa)} g_\kappa(X_s) ds \tag{6.3}$$

$$\delta_t^{(2)}(\kappa) := -\frac{i}{\kappa} \int_0^t a(s)^2 \left( \frac{e^{2i\theta_s(\kappa)}}{2} - 1 \right) e^{2i\theta_s(\kappa)} g_\kappa(X_s) F(X_s) ds.$$

Lemma 6.2(1) is proved.

(2) It is easy to see

$$\lim_{t \rightarrow \infty} \delta_t^{(1)}(\kappa) = \delta_\infty^{(1)}(\kappa), \text{ a.s..}$$

To see the convergence of  $\delta_t^{(2)}(\kappa)$  we write

$$\begin{aligned} \delta_t^{(2)}(\kappa) &= -\frac{i}{2\kappa} D_t^{(4)}(\kappa) + \frac{i}{\kappa} D_t^{(2)}(\kappa) \\ D_t^{(\beta)}(\kappa) &:= \int_0^t a(s)^2 e^{i\beta\theta_s(\kappa)} F(X_s) g_\kappa(X_s) ds, \quad \beta = 2, 4. \end{aligned} \quad (6.4)$$

We use Lemma 6.1(1) to decompose  $D_t^{(\beta)}(\kappa)$  into martingale part and the remainder : Setting  $h_{\kappa,\beta} = R_{\beta\kappa}(Fg_\kappa)$  and  $\widetilde{M}_s^{(\beta)}(\kappa) = M_s(Fg_\kappa, \beta\kappa)$ , we have

$$\begin{aligned} D_t^{(\beta)}(\kappa) &= I_t^{(\beta)}(\kappa) + N_t^{(\beta)}(\kappa) \\ I_t^{(\beta)}(\kappa) &:= [a(s)^2 e^{i\beta\theta_s(\kappa)} h_{\kappa,\beta}(X_s)]_0^t - \int_0^t (a(s)^2)' e^{i\beta\theta_s(\kappa)} h_{\kappa,\beta}(X_s) ds \\ &\quad - \frac{i\beta}{2\kappa} \int_0^t a(s)^3 \operatorname{Re}(e^{2i\theta_s(\kappa)} - 1) e^{i\beta\theta_s(\kappa)} F(X_s) h_{\kappa,\beta}(X_s) ds \\ N_t^{(\beta)}(\kappa) &:= \int_0^t a(s)^2 e^{i\beta\theta_s(\kappa)} d\widetilde{M}_s^{(\beta)}(\kappa). \end{aligned} \quad (6.5)$$

$I_t^{(\beta)}(\kappa)$  is easily seen to be convergent :  $\lim_{t \rightarrow \infty} I_t^{(\beta)}(\kappa) = I_\infty^{(\beta)}(\kappa)$ , a.s.. Since

$$|\langle N^{(\beta)}, N^{(\beta)} \rangle_t|, \quad |\langle N^{(\beta)}, \overline{N^{(\beta)}} \rangle_t| \leq (\text{const.}) \int_0^t a^4(s) ds < \infty$$

$\operatorname{Re} N$ ,  $\operatorname{Im} N$  can be represented by the time-change of a Brownian motion and thus have limit a.s..

(3) We consider  $\delta_t^{(1)}(\kappa)$ ,  $\delta_t^{(2)}(\kappa)$  separately. For  $\delta_t^{(1)}(\kappa)$ , we have

$$\begin{aligned} &\delta_{nt}^{(1)}(\kappa + \frac{c}{n}) - \delta_{nt}^{(1)}(\kappa) \\ &= a(nt) (e^{2i\theta_{nt}(\kappa + \frac{c}{n})} - e^{2i\theta_{nt}(\kappa)}) g_{\kappa + \frac{c}{n}}(X_{nt}) \\ &\quad - \int_0^{nt} a'(s) (e^{2i\theta_s(\kappa + \frac{c}{n})} - e^{2i\theta_s(\kappa)}) g_{\kappa + \frac{c}{n}}(X_s) ds + O(n^{-1}) \end{aligned}$$

by (6.3). The second term is  $o(1)$  as  $n \rightarrow \infty$  due to the Lebesgue's dominated convergence theorem. Thus the following equation will give us  $\mathbf{E}[|\delta_{nt}^{(1)}(\kappa +$

$$\frac{c}{n}) - \delta_{nt}^{(1)}(\kappa)|^2] \xrightarrow{n \rightarrow \infty} 0.$$

$$\max_{0 \leq t \leq T} a(nt) |e^{2i\theta_{nt}(\kappa + \frac{c}{n})} - e^{2i\theta_{nt}(\kappa)}| \xrightarrow{n \rightarrow \infty} 0. \quad (6.6)$$

Take any  $M > 0$ .

$$\begin{aligned} & \max_{0 \leq t \leq T} a(nt) |e^{2i\theta_{nt}(\kappa + \frac{c}{n})} - e^{2i\theta_{nt}(\kappa)}| \\ &= \max_{0 \leq t \leq M/n} a(nt) |e^{2i\theta_{nt}(\kappa + \frac{c}{n})} - e^{2i\theta_{nt}(\kappa)}| \vee \max_{M/n \leq t \leq T} a(nt) |e^{2i\theta_{nt}(\kappa + \frac{c}{n})} - e^{2i\theta_{nt}(\kappa)}| \\ &\leq C \max_{0 \leq t \leq M} |e^{2i\theta_t(\kappa + \frac{c}{n})} - e^{2i\theta_t(\kappa)}| \vee 2a(M). \end{aligned}$$

By (2.1)-(2.3) we have

$$\max_{0 \leq t \leq M} |e^{2i\theta_t(\kappa + \frac{c}{n})} - e^{2i\theta_t(\kappa)}| \leq \frac{C_M}{n}. \quad (6.7)$$

for some positive constant  $C_M$  depending on  $M$ . Hence

$$\limsup_{n \rightarrow \infty} \max_{0 \leq t \leq T} a(nt) |e^{2i\theta_{nt}(\kappa + \frac{c}{n})} - e^{2i\theta_{nt}(\kappa)}| \leq 2a(M).$$

Since  $M$  is arbitrary, we obtain (6.6).

Similar argument shows  $\max_{0 \leq t \leq T} |I_{nt}^{(\beta)}(\kappa + \frac{c}{n}) - I_{nt}^{(\beta)}(\kappa)| \rightarrow 0$  so that we have only to show

$$\mathbf{E} \left[ \max_{0 \leq t \leq T} \left| N_{nt}^{(\beta)}(\kappa + \frac{c}{n}) - N_{nt}^{(\beta)}(\kappa) \right|^2 \right] \xrightarrow{n \rightarrow \infty} 0, \quad \beta = 2, 4$$

to finish the proof of Lemma 6.2(3). By the martingale inequality,

$$\begin{aligned} & \mathbf{E} \left[ \max_{0 \leq t \leq T} \left| N_{nt}^{(\beta)}(\kappa + \frac{c}{n}) - N_{nt}^{(\beta)}(\kappa) \right|^2 \right] \\ &\leq C \mathbf{E} \left[ \int_0^{nt} a(s)^4 \left[ e^{i\beta\theta_s(\kappa + \frac{c}{n})} h_{\beta, \kappa + \frac{c}{n}} - e^{i\beta\theta_s(\kappa)} h_{\beta, \kappa}, \overline{e^{i\beta\theta_s(\kappa + \frac{c}{n})} h_{\beta, \kappa + \frac{c}{n}} - e^{i\beta\theta_s(\kappa)} h_{\beta, \kappa}} \right] ds \right] \end{aligned}$$

which converges to 0 due to the fact that  $\int_0^\infty a(s)^4 ds < \infty$  and Lebesgue's theorem.  $\square$

We next set

$$V_t^{(n)}(c) := Y_{nt}(\kappa + \frac{c}{n}) - Y_{nt}(\kappa)$$

and assume in what follows

$$a(t) = t^{-1/2}(1 + o(1)).$$

**Lemma 6.3**

$$\Psi_t^{(n)}(c) = 2ct + \operatorname{Re} \epsilon_t^{(n)} + \frac{1}{\kappa} \operatorname{Re} V_t^{(n)}(c) + \frac{1}{\kappa} \operatorname{Re} \left( \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right) \quad (6.8)$$

for some  $\epsilon_t^{(n)}$  satisfying

$$|\epsilon_t^{(n)}| \leq Ct + C\sqrt{\frac{t}{n}}.$$

*Proof.* We compute the third term of (6.1) by using Lemma 6.2

$$\begin{aligned} & \int_0^{nt} (e^{2i\theta_s(\kappa + \frac{c}{n})} - e^{2i\theta_s(\kappa)}) a(s) F(X_s) ds \\ &= \frac{i}{2} \cdot \frac{\frac{c}{n}}{\kappa(\kappa + \frac{c}{n})} \int_0^{nt} a(s)^2 g_{\kappa + \frac{c}{n}}(X_s) F(X_s) ds \\ &+ \frac{i}{2\kappa} \int_0^{nt} a(s)^2 (g_\kappa(X_s) - g_{\kappa + \frac{c}{n}}(X_s)) F(X_s) ds \\ &+ V_t^{(n)}(c) + \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa). \end{aligned}$$

Therefore

$$\Psi_t^{(n)}(c) = 2ct + \operatorname{Re} \epsilon_t^{(n)} + \frac{1}{\kappa} \operatorname{Re} V_t^{(n)}(c) + \frac{1}{\kappa} \operatorname{Re} \left( \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right)$$

where

$$\begin{aligned} \epsilon_t^{(n)} &:= -\frac{\frac{c}{n}}{\kappa(\kappa + \frac{c}{n})} \int_0^{nt} (e^{2i\theta_s(\kappa)} - 1) a(s) F(X_s) ds \\ &+ \frac{1}{\kappa} \left\{ \frac{i}{2} \cdot \frac{\frac{c}{n}}{\kappa(\kappa + \frac{c}{n})} \int_0^{nt} a(s)^2 g_{\kappa + \frac{c}{n}}(X_s) F(X_s) ds \right. \\ &\left. + \frac{i}{2\kappa} \int_0^{nt} a(s)^2 (g_\kappa(X_s) - g_{\kappa + \frac{c}{n}}(X_s)) F(X_s) ds \right\}. \end{aligned}$$

It then suffices to see

$$|\epsilon_t^{(n)}| \leq \frac{C}{n} \int_0^{nt} a(s) ds \leq C\sqrt{\frac{t}{n}} + Ct.$$

□

**Lemma 6.4**

$$\mathbf{E}[|\Psi_t^{(n)}(c)|] \leq C \left( t + \sqrt{\frac{t}{n}} + \frac{1}{\sqrt{n}} \right), \quad t \geq 0, \quad n \in \mathbf{N}.$$

*Proof.* We decompose  $\delta_t(\kappa)$  as is done in (6.2) to estimate  $\delta_t(\kappa)$  further.

$$\delta_t(\kappa) = \delta_t^{(1)}(\kappa) + \delta_t^{(2)}(\kappa). \quad (6.9)$$

Let

$$\Lambda_t^{(n)}(c) := e^{2i\theta_{nt}(\kappa + \frac{c}{n})} - e^{2i\theta_{nt}(\kappa)}$$

then

$$\begin{aligned} \delta_{nt}^{(1)}\left(\kappa + \frac{c}{n}\right) - \delta_{nt}^{(1)}(\kappa) &= \Lambda_t^{(n)}(c) a(nt) g_{\kappa + \frac{c}{n}}(X_{nt}) \\ &\quad - \int_0^{nt} a'(s) g_{\kappa + \frac{c}{n}}(X_s) \Lambda_{s/n}^{(n)}(c) ds + O(n^{-1}). \end{aligned} \quad (6.10)$$

$\delta_t^{(2)}$  is also decomposed, as in (6.4), (6.5)

$$\delta_t^{(2)}(\kappa) = -\frac{i}{2\kappa} D_t^{(4)}(\kappa) + \frac{i}{\kappa} D_t^{(2)}(\kappa) \quad (6.11)$$

$$D_t^{(\beta)}(\kappa) = I_t^{(\beta)}(\kappa) + N_t^{(\beta)}(\kappa), \quad \beta = 2, 4. \quad (6.12)$$

The  $I_{nt}^{(\beta)}$ -term can be written as

$$\begin{aligned} I_{nt}^{(\beta)}\left(\kappa + \frac{c}{n}\right) - I_{nt}^{(\beta)}(\kappa) &= a(nt)^2 h_{\kappa,n}^{(\beta)}(nt) \Lambda_t^{(n)}(c) \\ &\quad - \int_0^{nt} (a(s)^2)' f_{\kappa,n}^{(\beta)}(s) \Lambda_{s/n}^{(n)}(c) ds - \int_0^{nt} a(s)^3 g_{\kappa,n}^{(\beta)}(s) \Lambda_{s/n}^{(n)}(c) ds \end{aligned} \quad (6.13)$$

for some bounded functions  $f_{\kappa,n}^{(\beta)}, g_{\kappa,n}^{(\beta)}, h_{\kappa,n}^{(\beta)}$ . Substituting (6.10)-(6.13) into (6.9) we have

$$\begin{aligned} \delta_{nt}\left(\kappa + \frac{c}{n}\right) - \delta_{nt}(\kappa) &= \Lambda_t^{(n)}(c) \left( a(nt) g_{\kappa + \frac{c}{n}}(X_{nt}) + a(nt)^2 h_{\kappa,n}(nt) \right) \\ &\quad + \int_0^{nt} \Lambda_{s/n}^{(n)}(c) b_{\kappa,n}(s) ds + N_{nt}\left(\kappa + \frac{c}{n}\right) - N_{nt}(\kappa) + O(n^{-1}) \end{aligned}$$

for some bounded functions  $h_{\kappa,n}, b_{\kappa,n}$  and a martingale  $N_t$ .  $b_{\kappa,n}(s)$  is a linear combination of  $a'(s) g_{\kappa + \frac{c}{n}}$ ,  $(a(s)^2)' f_{\kappa,n}^{(\beta)}$ , and  $a(s)^3 g_{\kappa,n}^{(\beta)}$ , so that it is integrable

:  $\int_0^\infty b_{\kappa,n}(s)ds < \infty$ . Taking expectations, the martingale terms vanish and it follows that

$$\begin{aligned} \mathbf{E} \left[ \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right] &= \mathbf{E} \left[ \Lambda_t^{(n)}(c) \left( a(nt)g_{\kappa+\frac{c}{n}}(X_{nt}) + a(nt)^2 h_{\kappa,n}(nt) \right) \right] \\ &\quad + \int_0^{nt} \mathbf{E} \left[ \Lambda_{s/n}^{(n)}(c) b_{\kappa,n}(s) \right] ds + O(n^{-1}). \end{aligned}$$

Therefore we can find a non-random function

$$b(s) = C(a'(s) + (a(s)^2)' + a(s)^3)$$

for some  $C > 0$  such that  $\int_0^\infty b(s)ds < \infty$  and

$$\left| \mathbf{E} \left[ \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right] \right| \leq Ca(nt) \mathbf{E}[|\Lambda_t^{(n)}(c)|] + \int_0^{nt} \mathbf{E}[|\Lambda_{s/n}^{(n)}(c)|] b(s)ds + \frac{C}{n}.$$

Here without loss of generality, we may suppose  $c \geq 0$ . We use  $\Psi_t^{(n)}(c) \geq 0$  for  $c \geq 0$  and take expectation in (6.8).

$$\begin{aligned} \mathbf{E}[|\Psi_t^{(n)}(c)|] &= \mathbf{E}[\Psi_t^{(n)}(c)] \\ &= 2ct + \mathbf{E}[Re \epsilon_t^{(n)}] + \frac{1}{\kappa} \mathbf{E} \left[ Re \left( \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right) \right] \\ &\leq Ct + C\sqrt{\frac{t}{n}} + Ca(nt) \mathbf{E}[|\Lambda_t^{(n)}(c)|] \\ &\quad + C \int_0^{nt} \mathbf{E} \left[ |\Lambda_{s/n}^{(n)}(c)| \right] b(s)ds + \frac{C}{n}. \end{aligned}$$

Let

$$\rho_n(t) := C \left( t + \sqrt{\frac{t}{n}} + \frac{1}{n} \right).$$

Since  $|\Lambda_t^{(n)}(c)| \leq |\Psi_t^{(n)}(c)|$ , we have

$$\begin{aligned} &\mathbf{E} \left[ |\Psi_t^{(n)}(c)| \right] \\ &\leq \rho_n(t) + Ca(nt) \mathbf{E} \left[ |\Psi_t^{(n)}(c)| \right] + C \int_0^{nt} \mathbf{E} \left[ |\Psi_{s/n}^{(n)}(c)| \right] b(s)ds. \end{aligned}$$

Fix  $M > 0$  arbitrary. We may suppose  $nt > M$  since otherwise Lemma 6.4 holds true by (6.7). (6.7) also implies

$$\begin{aligned} \int_0^M \mathbf{E} \left[ \left| \Psi_{s/n}^{(n)} \right| \right] b(s) ds &\leq 2 \int_0^M \mathbf{E} \left[ \max_{0 \leq s \leq M} \left| \theta_s(\kappa + \frac{c}{n}) - \theta_s(\kappa) \right| \right] b(s) ds \\ &\leq \frac{C}{n} \end{aligned}$$

which gives us

$$\begin{aligned} &\mathbf{E} \left[ \left| \Psi_t^{(n)}(c) \right| \right] \\ &\leq \rho_n(t) + Ca(M) \mathbf{E} \left[ \left| \Psi_t^{(n)}(c) \right| \right] + C \int_M^{nt} \mathbf{E} \left[ \left| \Psi_{s/n}^{(n)}(c) \right| \right] b(s) ds + \frac{C}{n}. \end{aligned}$$

Take  $M$  large enough such that  $Ca(M) < 1$  and renew the positive constant  $C$  in the definition of  $\rho_n(t)$ . Then we have

$$\mathbf{E} \left[ \left| \Psi_t^{(n)}(c) \right| \right] \leq \rho_n(t) + C \int_{M/n}^t \mathbf{E} \left[ \left| \Psi_s^{(n)}(c) \right| \right] nb(ns) ds.$$

By Grownwall's inequality,

$$\begin{aligned} &\mathbf{E} \left[ \left| \Psi_t^{(n)}(c) \right| \right] \\ &\leq \rho_n(t) + C \int_{M/n}^t \rho_n(s) nb(ns) \exp \left( C \int_s^t nb(nu) du \right) ds. \end{aligned}$$

Since  $b$  is integrable,  $\exp \left( C \int_s^t nb(nu) du \right)$  is bounded so that

$$\mathbf{E} \left[ \left| \Psi_t^{(n)}(c) \right| \right] \leq \rho_n(t) + C \int_{M/n}^t \rho_n(s) nb(ns) ds. \quad (6.14)$$

Substituting

$$\begin{aligned} \int_{M/n}^t \rho_n(s) nb(ns) ds &= C \int_M^{nt} \left( \frac{s}{n} + \sqrt{\frac{s}{n^2}} + \frac{1}{n} \right) b(s) ds \\ &\leq \frac{C}{\sqrt{n}} \end{aligned}$$

into (6.14) yields the conclusion.  $\square$



**Lemma 6.5** *For  $t > 0$ , we have*

$$\mathbf{E}[\langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t] \leq C \left( t + \sqrt{\frac{t}{n}} + \frac{\log(nt)}{\sqrt{n}} \right).$$

*In particular,  $\sup_n \mathbf{E}[\langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t] < \infty$ .*

*Proof.* By Lemma 6.1(2) we have

$$\begin{aligned} \langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t &= \int_0^{nt} a(s)^2 \left| e^{2i(\theta_s(\kappa + \frac{c}{n}) - \theta_s(\kappa))} - 1 \right|^2 [g_\kappa, \overline{g_\kappa}](X_s) ds \\ &= \langle [g_\kappa, \overline{g_\kappa}] \rangle \int_0^{nt} a(s)^2 \left| e^{2i(\theta_s(\kappa + \frac{c}{n}) - \theta_s(\kappa))} - 1 \right|^2 ds + O(n^{-\frac{1}{2}}). \end{aligned}$$

We take expectations and use Lemma 6.4.

$$\begin{aligned} &\mathbf{E}[\langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t] \\ &= Cn \int_0^t a(ns)^2 \mathbf{E} \left[ \left| e^{i\Psi_s^{(n)}(c)} - 1 \right|^2 \right] ds + O(n^{-\frac{1}{2}}) \\ &\leq Cn \int_0^t a(ns)^2 \mathbf{E} \left[ |\Psi_s^{(n)}(c)| \right] ds + O(n^{-\frac{1}{2}}) \\ &\leq Cn \int_0^t a(ns)^2 \left( s + \sqrt{\frac{s}{n}} + \frac{1}{\sqrt{n}} \right) ds + O(n^{-\frac{1}{2}}) \\ &\leq C \left( t + \sqrt{\frac{t}{n}} + \frac{\log(nt)}{\sqrt{n}} \right). \end{aligned}$$

□

**Lemma 6.6** *For each  $c > 0$ ,  $T > 0$  fixed we have*

$$\begin{aligned} &\mathbf{E} \left[ \sup_{0 \leq t \leq T} \Psi_t^{(n)}(c) \right] \\ &\leq C \left( T + \sqrt{\frac{T}{n}} \right) + C \left( T + \sqrt{\frac{T}{n}} + \frac{\log(nT)}{\sqrt{n}} \right)^{1/2} \\ &\quad + C \mathbf{E} \left[ \max_{0 \leq t \leq T} \left| \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right| \right]. \end{aligned}$$

*Proof.* We estimate the third term of (6.8) by the martingale inequality and use Lemma 6.5 :

$$\begin{aligned} \mathbf{E} \left[ \sup_{0 \leq t \leq T} |V_t^{(n)}(\kappa)| \right] &\leq C \mathbf{E} \left[ |V_T^{(n)}(\kappa)|^2 \right]^{1/2} \\ &\leq C \left( T + \sqrt{\frac{T}{n}} + \frac{\log(nT)}{\sqrt{n}} \right)^{1/2}. \end{aligned}$$

□

**Lemma 6.7** *For each  $0 < t_0 < t_1 < \infty$ , we can find  $C = C(t_0, t_1)$  such that for large  $n$ , we have*

$$\mathbf{E} \left[ \left| V_t^{(n)}(c) - V_s^{(n)}(c) \right|^4 \right] \leq C(t-s)^2$$

for any  $s, t \in [t_0, t_1]$ .

*Proof.* By martingale inequality,

$$\begin{aligned} \mathbf{E} \left[ \left| V_t^{(n)}(c) - V_s^{(n)}(c) \right|^4 \right] &\leq C \mathbf{E} \left[ \left| V_t^{(n)}(c) - V_s^{(n)}(c) \right|^2 \right]^2 \\ &\leq C \mathbf{E} \left[ \int_{ns}^{nt} a(u)^2 \left[ e^{2i\theta_s(\kappa + \frac{c}{n})} g_{\kappa + \frac{c}{n}} - e^{2i\theta_s(\kappa)} g_{\kappa}, \overline{e^{2i\theta_s(\kappa + \frac{c}{n})} g_{\kappa + \frac{c}{n}} - e^{2i\theta_s(\kappa)} g_{\kappa}} \right] (X_s) ds \right]^2 \\ &\leq C \left( \int_{ns}^{nt} a(u)^2 du \right)^2. \end{aligned}$$

We can find  $N = N(t_0)$  such that for  $n \geq N$

$$C \left( \int_{ns}^{nt} a(u)^2 du \right)^2 \leq C \log \left( 1 + \frac{t-s}{t_0} \right)^2 \leq C(t-s)^2.$$

□

### 6.3 Tightness of $\Psi$

**Lemma 6.8** *For any  $c = (c_1, c_2, \dots, c_m) \in \mathbf{R}^m$ , the sequence of  $\mathbf{R}^m$ -valued process  $\{\Psi_t^{(n)}(c)\}_{n \geq 1} = \{(\Psi_t^{(n)}(c_1), \dots, \Psi_t^{(n)}(c_m))\}_{n \geq 1}$  is tight as a family in  $C([0, T] \rightarrow \mathbf{R}^m)$ .*

*Proof.* It is sufficient to show

$$(1) \lim_{A \rightarrow \infty} \sup_n \mathbf{P}(|\Psi_t^{(n)}(c)| \geq A) = 0$$

$$(2) \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbf{P} \left( \sup_{0 \leq s, t \leq T, |t-s| < \delta} |\Psi_t^{(n)}(c) - \Psi_s^{(n)}(c)| > \rho \right) = 0, \quad T, \rho > 0.$$

(1) follows from Lemma 6.4. To prove (2), we fix  $M > 0$  arbitrary and decompose

$$\begin{aligned} & \mathbf{P} \left( \sup_{0 \leq s, t \leq T, |t-s| < \delta} |\Psi_t^{(n)}(c) - \Psi_s^{(n)}(c)| > \rho \right) \\ & \leq \mathbf{P} \left( \sup_{0 \leq s, t \leq M, |t-s| < \delta} |\Psi_t^{(n)}(c) - \Psi_s^{(n)}(c)| > \rho \right) \\ & \quad + \mathbf{P} \left( \sup_{M \leq s, t \leq T, |t-s| < \delta} |\Psi_t^{(n)}(c) - \Psi_s^{(n)}(c)| > \rho \right) \\ & =: I + II. \end{aligned}$$

Since  $\Psi_0^{(n)}(c) = 0$  we have

$$I \leq \mathbf{P} \left( \sup_{t \leq M} |\Psi_t^{(n)}(c)| > \frac{\rho}{2} \right) + \mathbf{P} \left( \sup_{s \leq M} |\Psi_s^{(n)}(c)| > \frac{\rho}{2} \right)$$

and we use Lemma 6.6

$$\begin{aligned} & \mathbf{P} \left( \sup_{t \leq M} |\Psi_t^{(n)}(c)| > \frac{\rho}{2} \right) \\ & \leq \frac{2}{\rho} \mathbf{E} \left[ \sup_{0 \leq t \leq M} |\Psi_t^{(n)}(c)| \right] \\ & \leq C \left( M + \sqrt{\frac{M}{n}} \right) + C \left( M + \sqrt{\frac{M}{n}} + \frac{\log(Mn)}{\sqrt{n}} \right)^{1/2} \\ & \quad + C \mathbf{E} \left[ \max_{0 \leq t \leq M} \left| \delta_{nt} \left( \kappa + \frac{c}{n} \right) - \delta_{nt}(\kappa) \right| \right]. \end{aligned}$$

By Lemma 6.2(3) the third term vanishes as  $n \rightarrow \infty$  and it holds that

$$\limsup_{n \rightarrow \infty} I \leq CM^{1/2}. \quad (6.15)$$

Thus following estimate will be sufficient

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} II = 0 \quad (6.16)$$

because (6.15), (6.16) would imply

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbf{P} \left( \sup_{0 \leq s, t \leq T, |t-s| < \delta} |\Psi_t^{(n)}(c) - \Psi_s^{(n)}(c)| > \rho \right) \leq CM^{1/2}$$

and the arbitrariness of  $M > 0$  will yield the conclusion. By Lemmas 6.2, 6.3, eq.(6.16) will follow from the following equation

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbf{P} \left( \sup_{M \leq t, s \leq T, |t-s| < \delta} \left| V_t^{(n)}(c) - V_s^{(n)}(c) \right| > \rho \right) = 0$$

which, in turn, follows from Lemma 6.7 and Kolmogorov's theorem.  $\square$

## 6.4 SDE satisfied by $\Psi$

In this subsection we show that  $\Psi^{(n)}$  has a limit  $\Psi$  which satisfies (5.1).

**Lemma 6.9** *For any  $c_1, \dots, c_m \in \mathbf{R}$ , the solution of the following martingale problem is unique:*

$$W_t(c_j) = \Psi_t(c_j) - 2c_j t, \quad j = 1, 2, \dots, m$$

are martingales whose variational process satisfy

$$\langle W(c_i), W(c_j) \rangle_t = D^2 \int_0^t s^{-1} \operatorname{Re} \{ (e^{i\Psi_s(c_i)} - 1) (e^{-i\Psi_s(c_j)} - 1) \} ds.$$

Moreover  $\Psi_t(c_j)$  can be characterized by the unique solution of the following SDE.

$$d\Psi_t = 2c_j dt + Dt^{-1/2} \operatorname{Re} [(e^{i\Psi_t(c_j)} - 1) dZ_t], \quad \Theta_0(c_j) = 0.$$

**Theorem 6.10** *For any  $c_1, \dots, c_m \in \mathbf{R}$ ,  $(\Psi_t^{(n)}(c_1), \dots, \Psi_t^{(n)}(c_m)) \xrightarrow{d} (\Psi_t(c_1), \dots, \Psi_t(c_m))$ .  $\Psi_t(c_j)$  satisfies*

$$d\Psi_t(c_j) = 2c_j dt + D \operatorname{Re} \left[ (e^{i\Psi_t(c_j)} - 1) \frac{dZ_t}{\sqrt{t}} \right] \quad (6.17)$$

$$\Psi_0(c_j) = 0, \quad D := \frac{\sqrt{\langle [g_\kappa, \overline{g_\kappa}] \rangle}}{\sqrt{2\kappa}}.$$

*Proof.* By Lemma 6.8, the sequence  $\{(\Psi_t^{(n)}(c_1), \dots, \Psi_t^{(n)}(c_m))\}_{n \geq 1}$  has a limit point  $(\Psi_t(c_1), \dots, \Psi_t(c_m))$ . Since Lemmas 6.2, 6.3 imply

$$\Psi_t^{(n)}(c) = 2ct + \frac{1}{\kappa} \text{Re } V_t^{(n)}(c) + o(1)$$

in probability, we study  $V_t^{(n)}(c)$ . By Lemma 6.1  $\langle V^{(n)}(c), V^{(n)}(c') \rangle_t \xrightarrow{n \rightarrow \infty} 0$  in mean square. Similarly,

$$\begin{aligned} & \langle V^{(n)}(c), \overline{V^{(n)}(c')} \rangle_t \\ &= \langle [g_\kappa, \overline{g_\kappa}] \rangle \int_0^{nt} a(s)^2 \left( e^{2i(\theta_s(\kappa + \frac{c}{n}) - \theta_s(\kappa))} - 1 \right) \overline{\left( e^{2i(\theta_s(\kappa + \frac{c'}{n}) - \theta_s(\kappa))} - 1 \right)} ds + o(1) \\ &= \langle [g_\kappa, \overline{g_\kappa}] \rangle \int_0^t na(nu)^2 \left( e^{i\Psi_u^{(n)}(c)} - 1 \right) \overline{\left( e^{i\Psi_u^{(n)}(c')} - 1 \right)} du + o(1). \end{aligned}$$

By Skorohod's theorem, we can suppose

$$\Psi_t^{(n)}(c) \rightarrow \Psi_t(c)$$

compact uniformly with respect to  $t$ . Hence for  $0 < s < t$ ,

$$\begin{aligned} & \langle V^{(n)}(c), \overline{V^{(n)}(c')} \rangle_t - \langle V^{(n)}(c), \overline{V^{(n)}(c')} \rangle_s \\ &= \langle [g_\kappa, \overline{g_\kappa}] \rangle \int_s^t na(nu)^2 \left( e^{i\Psi_u^{(n)}(c)} - 1 \right) \overline{\left( e^{i\Psi_u^{(n)}(c')} - 1 \right)} du + o(1) \\ &\xrightarrow{n \rightarrow \infty} \langle [g_\kappa, \overline{g_\kappa}] \rangle \int_s^t u^{-1} \left( e^{i\Psi_u(c)} - 1 \right) \overline{\left( e^{i\Psi_u(c')} - 1 \right)} du. \end{aligned}$$

On the other hand by Lemma 6.4 we have

$$\int_0^t \mathbf{E} \left[ |e^{i\Psi_s(c)} - 1|^2 \right] \frac{ds}{s} \leq C \int_0^t \mathbf{E} [|\Psi_s(c)|] \frac{ds}{s} < \infty$$

so that  $V_t(c) = \lim_{n \rightarrow \infty} V_t^{(n)}(c)$  is a square integrable continuous martingale whose variational process satisfy

$$\begin{aligned} & \langle V(c), V(c') \rangle_t = 0 \\ & \langle V(c), \overline{V(c')} \rangle_t = \langle [g_\kappa, \overline{g_\kappa}] \rangle \int_0^t \left( e^{i\Psi_s(c)} - 1 \right) \overline{\left( e^{i\Psi_s(c')} - 1 \right)} \frac{ds}{s}. \end{aligned}$$

Therefore

$$W_t(c) = \Psi_t(c) - 2ct = \frac{1}{\kappa} \text{Re } V_t(c)$$

is a square integrable continuous martingale whose variational process is equal to

$$\langle W(c), W(c') \rangle_t = \frac{\langle [g_\kappa, \overline{g_\kappa}] \rangle}{2\kappa^2} \int_0^t \text{Re} \left[ (e^{i\Psi_s(c)} - 1)(e^{-i\Psi_s(c')} - 1) \frac{ds}{s} \right].$$

Lemma 6.9 yields the conclusion.  $\square$

**Lemma 6.11** *For a.s.,  $\Psi_t(c)$  is continuous on  $[0, \infty) \times \mathbf{R}$  and is increasing with respect to  $c$ .*

*Proof.* We first show the following inequality : for  $p > 1$  sufficiently close to 1,

$$\mathbf{E}[|\Psi_t(c_1) - \Psi_t(c_2)|^p] \leq \frac{2^p(c_1 - c_2)^p}{1 - \frac{1}{2}(p-1)D^2} t^p. \quad (6.18)$$

Hence by Kolmogorov's theorem, for any fixed  $t > 0$ ,  $\Psi_t(c)$  has a continuous version with respect to  $c \in \mathbf{R}$  a.s.. We first note that  $\Psi_t(c)$  satisfies

$$d\Psi_t(c) = 2cdt + \frac{D}{2\sqrt{t}} \{ (e^{i\Psi_t} + e^{-i\Psi_t} - 2)dB_t^1 + i(e^{i\Psi_t} - e^{-i\Psi_t})dB_t^2 \}.$$

Here we note that if  $c_1 > c_2$  then  $\Psi_t(c_1) > \Psi_t(c_2)$  by the comparison theorem of SDE which proves the desired monotonicity of  $\Psi_t(c)$ . We set

$$\begin{aligned} \Gamma_t &:= \Psi_t(c_1) - \Psi_t(c_2) \\ \Xi_t &:= e^{i\Psi_t(c_1)} - e^{i\Psi_t(c_2)}. \end{aligned}$$

For  $c_1 > c_2$ , we see

$$d\Gamma_t = 2(c_1 - c_2)dt + \frac{D}{2\sqrt{t}} \{ (\Xi_t + \overline{\Xi}_t)dB_t^1 + i(\Xi_t - \overline{\Xi}_t)dB_t^2 \}.$$

Hence

$$(d\Gamma_t)^2 = \frac{D^2}{4t} \{ (\Xi_t + \overline{\Xi}_t)^2 - (\Xi_t - \overline{\Xi}_t)^2 \} dt = \frac{D^2}{t} |\Xi_t|^2 dt.$$

Then for  $p > 1$

$$\begin{aligned} d\Gamma_t^p &= p\Gamma_t^{p-1}dt + \frac{p(p-1)}{2}\Gamma_t^{p-2}(d\Gamma_t)^2 \\ &= (c_1 - c_2)p\Gamma_t^{p-1}dt + \frac{p(p-1)}{2}\Gamma_t^{p-2}\frac{D^2}{t}|\Xi_t|^2dt \\ &\quad + p\Gamma_t^{p-1}\frac{D}{2\sqrt{t}}\{(\Xi_t + \overline{\Xi}_t)dB_t^1 + i(\Xi_t - \overline{\Xi}_t)dB_t^2\}. \end{aligned}$$

Taking the expectation we have

$$\mathbf{E}[\Gamma_t^p] = 2(c_1 - c_2)p \int_0^t \mathbf{E}[\Gamma_s^{p-1}]ds + \frac{p(p-1)}{2}D^2 \int_0^t \mathbf{E}[\Gamma_s^{p-2}|\Xi_s|^2]\frac{ds}{s}. \quad (6.19)$$

We can find

$$|\Xi_t|^2 \leq C\Gamma_t^\gamma, \quad 0 < \gamma < 2$$

for some positive constant  $C$  and some  $0 < \gamma < 2$ . Then

$$\int_0^t \mathbf{E}[\Gamma_s^{p-2}|\Xi_s|^2]\frac{ds}{s} \leq C \int_0^t \mathbf{E}[\Gamma_s^{p-2+\gamma}]\frac{ds}{s}.$$

We use  $\mathbf{E}[|X|^r] \leq \mathbf{E}[|X|]^r$  for  $r \leq 1$  and the fact that  $\mathbf{E}[\Psi_t(c)] = 2ct$ . Assuming  $p-1 \leq 1$  and  $p-2+\gamma \leq 1$  yields

$$\begin{aligned} \mathbf{E}[\Gamma_t^p] &\leq 2(c_1 - c_2)p \int_0^t \mathbf{E}[\Gamma_s]^{p-1}ds + C \int_0^t \mathbf{E}[\Gamma_s]^{p-2+\gamma}\frac{ds}{s} \\ &= 2^p(c_1 - c_2)^p t^p + C(c_1 - c_2)^{p-2+\gamma} t^{p-2+\gamma} \end{aligned}$$

so that for  $0 \leq t \leq T$  we have

$$f(t) := \mathbf{E}[\Gamma_t^p] \leq C_T t^{p-2+\gamma}$$

and hence

$$h(t) := \int_0^t \frac{f(s)}{s} ds \leq C_T t^{p-2+\gamma}.$$

Thus for any  $p > 1$  sufficiently close to 1, we take  $\gamma$  satisfying  $1 + (p-1)(\frac{p}{2}D^2 - 1) < \gamma \leq 3 - p$  so that

$$h(t) \leq C t^{\frac{p}{2}(p-1)D^2 + \delta} \quad (6.20)$$

for some  $\delta > 0$ .

On the other hand by using  $|\Xi_s|^2 \leq \Gamma_s^2$  in (6.19) we have

$$\begin{aligned} \mathbf{E}[\Gamma_t^p] &\leq 2(c_1 - c_2)p \int_0^t 2^{p-1}(c_1 - c_2)^{p-1} s^{p-1} ds + \frac{p}{2}(p-1)D^2 \int_0^t \mathbf{E}[\Gamma_s^p] \frac{ds}{s} \\ &= 2^p(c_1 - c_2)^p t^p + \frac{p}{2}(p-1)D^2 \int_0^t \mathbf{E}[\Gamma_s^p] \frac{ds}{s}. \end{aligned}$$

Hence if  $\frac{1}{2}(p-1)D^2 < 1$ , (6.20) and a Grownwall type argument give the desired inequality (6.18).

Having established the continuity of  $\Psi_{t_0}(c)$  with respect to  $c$ , the joint continuity of  $\Psi_t(c)$  on  $[t_0, \infty) \times \mathbf{R}$  is valid due to the absence of singularity in this time domain. The continuity of  $\Psi_t(c)$  at  $t = 0$  follows from the monotonicity of  $\Psi_t(c)$  with respect to  $c$ .  $\square$

**Theorem 6.12** *The limit process  $\{\Psi_t(c)\}_{t \geq 0, c \in \mathbf{R}}$  satisfies the following two properties :*

(1) *The process has invariance*

$$\{\Psi_t(c)\}_{t \geq 0, c \in \mathbf{R}} \stackrel{law}{=} \{\Psi_t(c + c_0) - \Psi_t(c_0)\}_{t \geq 0, c \in \mathbf{R}}$$

for any  $c_0 \in \mathbf{R}$ .

(2) *For each fixed  $c$  there exists a 1-D Brownian motion  $\{B_t(c)\}$  such that*

$$\frac{\partial \Psi_t(c)}{\partial c} = 2 \int_0^t \exp \left( D \int_s^t u^{-1/2} dB_u(c) \right) ds$$

$\{B_t(c)\}$  are a family of martingales satisfying

$$\langle B.(c), B.(c') \rangle_t = \int_0^t \cos(\Psi_s(c) - \Psi_s(c')) ds.$$

*Proof.* (1) For any fixed  $c_0 \in \mathbf{R}$ , letting  $\Phi_t(c) = \Psi_t(c + c_0) - \Psi_t(c_0)$  yields

$$d\Phi_t(c) = 2cdt + DRe[(e^{i\Psi_t(c+c_0)} - e^{i\Psi_t(c_0)}) t^{-1/2} dZ_t].$$

Here noting

$$\tilde{Z}_t = \int_0^t e^{i\Psi_s(c_0)} dZ_s$$



is a complex Brownian motion, we see

$$d\Phi_t(c) = 2cdt + DRe[(e^{i\Phi_t(c)} - 1)t^{-1/2}d\tilde{Z}_t], \quad \Phi_0(c) = 0$$

hence the uniqueness of the solutions gives us

$$\{\Phi_t(c)\}_{t \geq 0, c \in \mathbf{R}} \stackrel{law}{=} \{\Psi_t(c)\}_{t \geq 0, c \in \mathbf{R}}.$$

(2) Set

$$\Phi_t(c) := \frac{\partial \Psi_t(c)}{\partial c}, \quad B_t(c) := Re \int_0^t ie^{i\Psi_s(c)} dZ_s.$$

Then, for a fixed  $c$ ,  $\{B_t(c)\}_{t \geq 0}$  is a 1D Brownian motion and

$$d\Phi_t(c) = 2dt + D\Phi_t(c)Re[ie^{i\Psi_t(c)}t^{-1/2}dZ_t] = 2dt + Dt^{-1/2}\Phi_t(c)dB_t(c)$$

holds, hence

$$\Phi_t(c) = 2 \int_0^t \exp \left( D \int_s^t u^{-1/2} dB_u(c) \right) ds.$$

The variational process for  $\{B_t(c)\}$  are

$$\begin{aligned} & \langle B_t(c), B_t(c') \rangle_t \\ &= \frac{1}{4} \left\langle \int_0^t ie^{i\Psi_s(c)} dZ_s - \int_0^t ie^{-i\Psi_s(c)} d\overline{Z}_s, \int_0^t ie^{i\Psi_s(c')} dZ_s - \int_0^t ie^{-i\Psi_s(c')} d\overline{Z}_s \right\rangle_t \\ &= \frac{1}{4} \int_0^t e^{i(\Psi_s(c') - \Psi_s(c))} 2ds + \frac{1}{4} \int_0^t e^{i(\Psi_s(c) - \Psi_s(c'))} 2ds \\ &= \int_0^t \cos(\Psi_s(c) - \Psi_s(c')) ds. \end{aligned}$$

□

## 7 Convergence of $\theta_t(\kappa) \bmod \pi$

**Proposition 7.1** *As  $t \rightarrow \infty$   $(2\theta_t(\kappa))_{2\pi\mathbf{Z}}$  converges to the uniform distribution on  $[0, 2\pi)$ .*

*Proof.* Letting

$$\xi_t(\kappa) := e^{2mi\tilde{\theta}_t(\kappa)}, \quad m \in \mathbf{Z}$$

it suffices to show

$$\mathbf{E}[\xi_t(\kappa)] \xrightarrow{t \rightarrow \infty} 0, \quad m \neq 0.$$

We omit the  $\kappa$ -dependence of  $\theta_t$ . By (2.2) we decompose

$$\begin{aligned} \xi_t &= 1 + \frac{mi}{2\kappa} \int_0^t e^{2i\kappa s + 2(m+1)i\tilde{\theta}_s} a(s) F(X_s) ds \\ &\quad + \frac{mi}{2\kappa} \int_0^t e^{-2i\kappa s + 2(m-1)i\tilde{\theta}_s} a(s) F(X_s) ds \\ &\quad - \frac{mi}{\kappa} \int_0^t e^{2mi\tilde{\theta}_s} a(s) F(X_s) ds \\ &=: 1 + I + II + III. \end{aligned}$$

We use Lemma 6.1(1) and decompose  $I$  further into “non-oscillating”-term + martingale-term + remainder :

$$\begin{aligned} I &= \frac{mi}{2\kappa} \left( -\frac{2i(m+1)}{4\kappa} \int_0^t a(s)^2 e^{2mi\tilde{\theta}_s} F(X_s) g_\kappa(X_s) ds \right. \\ &\quad \left. + \int_0^t a(s) e^{2i\kappa s} e^{2i(m+1)\tilde{\theta}_s} dM_s(\kappa) + \delta_{1,1}(t) \right). \end{aligned} \quad (7.1)$$

where

$$\begin{aligned} &\delta_{1,1}(t) \\ &:= \left[ a(s) e^{2i(m+1)\tilde{\theta}_s} e^{2i\kappa s} g_\kappa(X_s) \right]_0^t \\ &\quad - \int_0^t a'(s) e^{2i(m+1)\tilde{\theta}_s} e^{2i\kappa s} g_\kappa(X_s) ds \\ &\quad - \frac{2i(m+1)}{2\kappa} \int_0^t a(s)^2 \left( \frac{e^{2(m+2)i\tilde{\theta}_s} e^{4i\kappa s}}{2} - e^{2(m+1)i\tilde{\theta}_s} e^{2i\kappa s} \right) F(X_s) g_\kappa(X_s) ds. \end{aligned}$$

We further compute the third term of  $\delta_{1,1}$  by Lemma 6.1(1) and see that  $\delta_{1,1}(t)$  has a limit as  $t \rightarrow \infty$ . Taking expectation, martingale term vanishes and we have

$$\mathbf{E}[\delta_{1,1}(t)] - \mathbf{E}[\delta_{1,1}(\infty)] = O(a(t)), \quad t \rightarrow \infty. \quad (7.2)$$

By Lemma 6.1(2), the first term of (7.1) satisfies

$$\int_0^t a(s)^2 e^{2mi\tilde{\theta}_s} F(X_s) g_\kappa(X_s) ds = \langle F g_\kappa \rangle \int_0^t a(s)^2 e^{2mi\tilde{\theta}_s} ds + \delta_{1,2}(t)$$

where  $\delta_{1,2}(t)$  has a limit as  $t \rightarrow \infty$  and satisfies the same estimate as (7.2). We substitute it into (7.1) and let  $\delta_1 = \delta_{1,1} + \delta_{1,2}$ . Then

$$I = \frac{mi}{2\kappa} \left( -\frac{2i(m+1)}{4\kappa} \langle Fg_\kappa \rangle \int_0^t a(s)^2 e^{2mi\tilde{\theta}_s} ds + \int_0^t a(s) e^{2i\kappa s} e^{2i(m+1)\tilde{\theta}_s} dM_s(\kappa) + \delta_1(t) \right). \quad (7.3)$$

Similarly, we have

$$\begin{aligned} II &= \frac{mi}{2\kappa} \left( -\frac{2(m-1)i}{4\kappa} \langle Fg_{-\kappa} \rangle \int_0^t a(s)^2 e^{2mi\tilde{\theta}_s} ds + \int_0^t a(s) e^{-2i\kappa s} e^{2i(m-1)\tilde{\theta}_s} dM_s(-\kappa) + \delta_2(t) \right) \\ III &= -\frac{mi}{\kappa} \left\{ \langle F \rangle \int_0^t a(s) e^{2mi\tilde{\theta}_s} ds + \frac{2mi}{2\kappa} \langle Fg \rangle \int_0^t a(s)^2 e^{2mi\tilde{\theta}_s} ds + \int_0^t a(s) e^{2mi\tilde{\theta}_s} dM_s + \delta_3(t) \right\}. \end{aligned}$$

To summarize,

$$\xi_t = 1 - \frac{mi}{\kappa} \langle F \rangle \int_0^t a(s) e^{2mi\tilde{\theta}_s} ds + \langle G \rangle \int_0^t a(s)^2 e^{2mi\tilde{\theta}_s} ds + N_t + \delta(t) \quad (7.4)$$

where

$$\begin{aligned} G &= \left( \frac{m(m+1)}{4\kappa^2} g_\kappa + \frac{m(m-1)}{4\kappa^2} g_{-\kappa} + \frac{m^2}{\kappa^2} g \right) F \\ N_t &= \frac{mi}{2\kappa} \int_0^t a(s) e^{2i\kappa s} e^{2i(m+1)\tilde{\theta}_s} dM_s(\kappa) \\ &\quad + \frac{mi}{2\kappa} \int_0^t a(s) e^{-2i\kappa s} e^{2i(m-1)\tilde{\theta}_s} dM_s(-\kappa) \\ &\quad - \frac{mi}{\kappa} \int_0^t a(s) e^{2mi\tilde{\theta}_s} dM_s \end{aligned}$$

where  $\delta(\infty) = \lim_{t \rightarrow \infty} \delta(t)$  exists a.s. and

$$\mathbf{E}[\delta(t)] - \mathbf{E}[\delta(\infty)] = O(a(t)), \quad t \rightarrow \infty.$$

Let  $\sigma_F(d\lambda)$  be the spectral measure of  $L$  with respect to  $F$ . Then by noting

$$Re\langle Fg_\kappa \rangle = Re\langle Fg_{-\kappa} \rangle = \int_{-\infty}^0 \frac{\lambda \sigma_F(d\lambda)}{\lambda^2 + 4\kappa^2} < 0, \quad Re\langle Fg \rangle = \int_{-\infty}^0 \frac{\sigma_F(d\lambda)}{\lambda} < 0$$

we have

$$-\gamma := Re\langle G \rangle < 0.$$

Set

$$\rho(t) := \mathbf{E}[\xi_t], \quad b(t) := -\frac{mi}{2\kappa} \langle F \rangle a(t) + \langle G \rangle a(t)^2.$$

Then (7.4) turns to

$$\rho(t) = 1 + \int_0^t b(s) \rho(s) ds + \mathbf{E}[\delta(t)]$$

and hence

$$\begin{aligned} \rho(t) &= \exp \left( \int_0^t b(u) du \right) + \mathbf{E}[\delta(t)] + \int_0^t \mathbf{E}[\delta(s)] b(s) \exp \left( \int_s^t b(u) du \right) ds \\ &= \exp \left( \int_0^t b(u) du \right) + \mathbf{E}[\delta(t)] + \mathbf{E}[\delta(\infty)] \int_0^t b(s) \exp \left( \int_s^t b(u) du \right) ds \\ &\quad + \int_0^t (\mathbf{E}[\delta(s)] - \mathbf{E}[\delta(\infty)]) b(s) \exp \left( \int_s^t b(u) du \right) ds \\ &=: I + II + III + IV. \end{aligned}$$

Noting  $Re b(t) = Re \langle G \rangle a(t)^2 = -\gamma a(t)^2$ , we compute  $I, III$

$$\begin{aligned} |I| &\leq \exp \left( \int_0^t Re b(s) ds \right) \leq C \exp \left( -\gamma \int_1^t \frac{1}{s} ds \right) \xrightarrow{t \rightarrow \infty} 0 \\ III &= \mathbf{E}[\delta(\infty)] \left( -1 + \exp \left( \int_0^t b(u) du \right) \right) \xrightarrow{t \rightarrow \infty} -\mathbf{E}[\delta(\infty)]. \end{aligned}$$

We further decompose  $IV$  :

$$\begin{aligned} |IV| &= \left| \int_0^t (\mathbf{E}[\delta(s)] - \mathbf{E}[\delta(\infty)]) b(s) \exp \left( \int_s^t b(u) du \right) ds \right| \\ &\leq C \left( \int_0^M + \int_M^t \right) a(s) |b(s)| \exp \left( Re \int_s^t b(u) du \right) ds \\ &=: IV_1 + IV_2. \end{aligned}$$

It is easy to see that  $IV_1 \xrightarrow{t \rightarrow \infty} 0$ . For  $IV_2$  we use  $\langle F \rangle = 0$  and compute, for large  $M$ ,

$$\begin{aligned} |IV_2| &\leq C \int_M^t a(s)^3 \exp \left( \int_s^t \operatorname{Re} b(u) du \right) ds \\ &\leq C \int_M^t s^{-3/2} \left( \frac{t}{s} \right)^{-\gamma} ds \\ &= \begin{cases} Ct^{-\gamma} \log \frac{t}{M} & (\gamma = \frac{1}{2}) \\ Ct^{-\gamma} \frac{t^{\gamma-\frac{1}{2}} - M^{\gamma-\frac{1}{2}}}{\gamma-\frac{1}{2}} & (\gamma \neq \frac{1}{2}) \end{cases} \xrightarrow{t \rightarrow \infty} 0. \end{aligned}$$

□

## 8 Limiting behavior of $\tilde{\theta}_t$

To study the limiting behavior of  $(2\tilde{\theta}_t)_{2\pi\mathbf{Z}}$  we set

$$\tilde{\xi}_t := e^{2i\tilde{\theta}_t(\kappa)}.$$

### 8.1 Estimate of integral equation

As in the proof of Proposition 7.1, we can show the following lemma.

**Lemma 8.1** *Let  $0 < t_0 < t$ . Then we have*

$$\begin{aligned} \tilde{\xi}_t &= \tilde{\xi}_{t_0} + \frac{1}{2\kappa^2} \langle F \cdot (g_\kappa + 2g) \rangle \int_{t_0}^t a(s)^2 e^{2i\tilde{\theta}_s} ds - \frac{i}{\kappa} \langle F \rangle \int_{t_0}^t a(s) e^{2i\tilde{\theta}_s} ds \\ &\quad + \frac{i}{2\kappa} \left( Y_t + \tilde{Y}_t - 2\hat{Y}_t \right) + O(a(t_0)), \quad t_0 \rightarrow \infty. \end{aligned}$$

$$\begin{aligned} \text{where } Y_t &:= \int_{t_0}^t a(s) e^{2i\kappa s + 4i\tilde{\theta}_s} dM_s(\kappa) \\ \tilde{Y}_t &:= \int_{t_0}^t a(s) e^{-2i\kappa s} dM_s(-\kappa) \\ \hat{Y}_t &:= \int_{t_0}^t a(s) e^{2i\tilde{\theta}_s} dM_s. \end{aligned}$$

The variational process of  $Y, \tilde{Y}$ , and  $\hat{Y}$  satisfy, as  $t_0 \rightarrow \infty$ ,

$$\begin{aligned}
\langle Y, Y \rangle_t &= O(a(t_0)) \\
\langle Y, \bar{Y} \rangle_t &= \langle [g_\kappa, \bar{g}_\kappa] \rangle \int_{t_0}^t a(s)^2 ds + O(a(t_0)) \\
\langle \tilde{Y}, \tilde{Y} \rangle_t &= O(a(t_0)) \\
\langle \tilde{Y}, \bar{\tilde{Y}} \rangle_t &= \langle [g_\kappa, \bar{g}_\kappa] \rangle \int_{t_0}^t a(s)^2 ds + O(a(t_0)) \\
\langle \hat{Y}, \hat{Y} \rangle_t &= \langle [g, g] \rangle \int_{t_0}^t a(s)^2 e^{4i\tilde{\theta}_s} ds + O(a(t_0)) \\
\langle \hat{Y}, \bar{\hat{Y}} \rangle_t &= \langle [g, g] \rangle \int_{t_0}^t a(s)^2 ds + O(a(t_0)).
\end{aligned}$$

## 8.2 Tightness of $\eta$

Let

$$\begin{aligned}
\eta_t^{(n)} &:= \xi_{nt} = e^{2i\tilde{\theta}_{nt}(\kappa)} \\
\mathbf{U} &:= \{z \in \mathbf{C} \mid |z| = 1\}.
\end{aligned}$$

**Lemma 8.2**  $\{\eta_t^{(n)}\}_{n \geq 1}$  is tight as a family in  $C((0, \infty) \rightarrow \mathbf{U})$ .

*Proof.* It suffices to show, for any  $t_0 > 0, \rho > 0$ ,

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbf{P} \left( \sup_{t_0 < s < t, t-s < \delta} |\eta_t^{(n)} - \eta_s^{(n)}| > \rho \right) = 0.$$

Noting  $\langle F \rangle = 0$ , Lemma 8.1 implies

$$\begin{aligned}
\tilde{\xi}_{nt} - \tilde{\xi}_{ns} &= \frac{1}{2\kappa^2} \langle F \cdot (g_\kappa + 2g) \rangle \int_{ns}^{nt} a(u)^2 e^{2i\tilde{\theta}_u} du \\
&\quad + \frac{i}{2\kappa} W_{t,s}^{(n)} + o(1)
\end{aligned} \tag{8.1}$$

$$\text{where } W_{t,s}^{(n)} := \left( Y_{nt} + \tilde{Y}_{nt} - 2\hat{Y}_{nt} \right) - \left( Y_{ns} + \tilde{Y}_{ns} - 2\hat{Y}_{ns} \right).$$

We note that  $W_{t,s}^{(n)}$  satisfies the estimate in Lemma 6.7 and the rest of the argument is the same as that of Lemma 6.8.  $\square$

### 8.3 Identification of $\eta_t$

Let  $\eta_t$  be a limit point of  $\eta_t^{(n)}$  which is uniformly distributed on  $\mathbf{U}$  for each fixed  $t > 0$  by Lemma 7.1. In this subsection we show that the distribution of the process  $\eta_t$  is uniquely determined.

**Lemma 8.3** (1) For any  $0 < t_0 < t$ ,

$$\lim_{n \rightarrow \infty} \mathbf{E}[e^{2mi(\tilde{\theta}_{nt} - \tilde{\theta}_{nt_0})} | \mathcal{F}_{nt_0}] = \left(\frac{t}{t_0}\right)^{\langle G_m \rangle}.$$

(2) For any  $0 < t_0 < t_1 < \dots < t_k$ , the family of random variables  $\{\eta_{t_0}, \eta_{t_1}/\eta_{t_0}, \dots, \eta_{t_k}/\eta_{t_{k-1}}\}$  are independent.

*Proof.* (1) Let  $m, m' \in \mathbf{Z}$ . By a argument similar to deduce (7.4), we have

$$\begin{aligned} e^{2mi(\tilde{\theta}_{nt} - \tilde{\theta}_{nt_0})} &= 1 + \langle G_m \rangle \int_{nt_0}^{nt} a(s)^2 e^{2mi(\tilde{\theta}_s - \tilde{\theta}_{nt_0})} ds \\ &\quad + N_{nt, nt_0} e^{-2mi\tilde{\theta}_{nt_0}} + \delta_n(t) e^{-2mi\tilde{\theta}_{nt_0}} \\ &= 1 + \langle G_m \rangle \int_{t_0}^t na(nu)^2 e^{2mi(\tilde{\theta}_{nu} - \tilde{\theta}_{nt_0})} du \\ &\quad + N_{nt, nt_0} e^{-2mi\tilde{\theta}_{nt_0}} + \delta_n(t) e^{-2mi\tilde{\theta}_{nt_0}} \end{aligned}$$

where

$$\begin{aligned} G_m &= \left( \frac{m(m+1)}{4\kappa^2} g_\kappa + \frac{m(m-1)}{4\kappa^2} g_{-\kappa} + \frac{m^2}{\kappa^2} g \right) F \\ N_{nt, nt_0} &= \frac{mi}{2\kappa} \int_{nt_0}^{nt} a(s) e^{2i\kappa s} e^{2i(m+1)\tilde{\theta}_s} dM_s(\kappa) \\ &\quad + \frac{mi}{2\kappa} \int_{nt_0}^{nt} a(s) e^{-2i\kappa s} e^{2i(m-1)\tilde{\theta}_s} dM_s(-\kappa) \\ &\quad - \frac{mi}{\kappa} \int_{nt_0}^{nt} a(s) e^{2mi\tilde{\theta}_s} dM_s \end{aligned}$$

$$\mathbf{E}[\delta_n(t) | \mathcal{F}_{nt_0}] \xrightarrow{n \rightarrow \infty} 0, \quad a.s..$$

Taking a conditional expectation and letting

$$\rho_n(t) := \mathbf{E}[e^{2mi(\tilde{\theta}_{nt} - \tilde{\theta}_{nt_0})} | \mathcal{F}_{nt_0}]$$

we have

$$\rho_n(t) = 1 + \langle G_m \rangle \int_{t_0}^t na(nu)^2 \rho_n(u) du + \mathbf{E}[\delta_n(t) | \mathcal{F}_{nt_0}] e^{-2mi\tilde{\theta}_{nt_0}}.$$

Therefore

$$\begin{aligned} \rho_n(t) &= \exp \left( \langle G_m \rangle \int_{t_0}^t na(nu)^2 du \right) + \mathbf{E}[\delta_n(t) | \mathcal{F}_{nt_0}] e^{-2mi\tilde{\theta}_{nt_0}} \\ &\quad + \int_{t_0}^t \mathbf{E}[\delta_n(s) | \mathcal{F}_{nt_0}] e^{-2mi\tilde{\theta}_{nt_0}} \langle G_m \rangle na(nu)^2 \exp \left( \langle G_m \rangle \int_s^t na(nu)^2 du \right) ds \\ &\xrightarrow{n \rightarrow \infty} \exp \left( \langle G_m \rangle \int_{t_0}^t \frac{du}{u} \right) = \left( \frac{t}{t_0} \right)^{\langle G_m \rangle}. \end{aligned}$$

(2) The required independence easily follows from (1) and the fact that  $e^{2i\tilde{\theta}_{nt}}$  converges to the uniform distribution on  $\mathbf{U}$  as  $n \rightarrow \infty$ . In fact, for  $n = 1$ ,

$$\begin{aligned} &\mathbf{E} \left[ \mathbf{E}[e^{2mi(\tilde{\theta}_{nt} - \tilde{\theta}_{nt_0})} | \mathcal{F}_{nt_0}] e^{2m'i\tilde{\theta}_{nt_0}} \right] \\ &= \mathbf{E} \left[ \left( \mathbf{E}[e^{2mi(\tilde{\theta}_{nt} - \tilde{\theta}_{nt_0})} | \mathcal{F}_{nt_0}] - \left( \frac{t}{t_0} \right)^{\langle G_m \rangle} \right) e^{2m'i\tilde{\theta}_{nt_0}} \right] + \mathbf{E} \left[ \left( \frac{t}{t_0} \right)^{\langle G_m \rangle} e^{2m'i\tilde{\theta}_{nt_0}} \right] \\ &\rightarrow \begin{cases} 0 & (m' \neq 0) \\ \left( \frac{t}{t_0} \right)^{\langle G_m \rangle} & (m' = 0) \end{cases} \end{aligned}$$

For  $n \geq 2$ , the proof is similar.  $\square$

**Lemma 8.4** *For each fixed  $t_0 > 0$   $\eta_t$  satisfies the following SDE on  $t \geq t_0$ :*

$$d\eta_t = C_1 \frac{\eta_t}{t} dt + C_2 \frac{\eta_t}{\sqrt{t}} dB_t, \quad (8.2)$$

$$\text{where } C_1 := \frac{\langle (g_\kappa + 2g)F \rangle}{2\kappa^2}, \quad C_2 := \frac{i}{2\kappa} \sqrt{\langle 2[g_\kappa, \overline{g_\kappa}] + 4[g, g] \rangle}.$$

*Proof.* Letting  $s = t_0 > 0$  in (8.1) yields, as  $n \rightarrow \infty$ ,

$$\begin{aligned} \int_{nt_0}^{nt} a(u)^2 e^{2i\tilde{\theta}_u} du &\rightarrow \int_{t_0}^t \frac{\eta_u}{u} du \\ \langle W_{\cdot, t_0}^{(n)}, W_{\cdot, t_0}^{(n)} \rangle_t &\rightarrow \langle 2[g_\kappa, \overline{g_\kappa}] + 4[g, g] \rangle \int_{t_0}^t \frac{\eta_u^2}{u} du \\ \langle W_{\cdot, t_0}^{(n)}, \overline{W_{\cdot, t_0}^{(n)}} \rangle_t &\rightarrow \langle 2[g_\kappa, \overline{g_\kappa}] + 4[g, g] \rangle \int_{t_0}^t \frac{du}{u}. \end{aligned}$$



We then proceed as in the proof of Theorem 6.10.  $\square$

**Remark 8.1**  $Z_t, B_t$  which appear in SDE's (6.17, 8.2) of  $\Psi$ ,  $\eta$  are not independent. In fact,

$$\begin{aligned} dW_t &= \sqrt{2\langle [g_\kappa, \overline{g_\kappa}] \rangle + 4[g, g]} \frac{\eta_t}{\sqrt{t}} dB_t \\ dV_t &= \sqrt{\langle [g_\kappa, \overline{g_\kappa}] \rangle} (e^{i\Psi_t(c)} - 1) \frac{dZ_t}{\sqrt{t}} \\ d\langle W, V \rangle &= \langle [g_\kappa, \overline{g_\kappa}] \rangle (e^{i\Psi_t(c)} - 1) \eta_t \frac{dt}{t} \\ d\langle W, \overline{V} \rangle &= \langle [g_\kappa, \overline{g_\kappa}] \rangle (e^{-i\Psi_t(c)} - 1) \eta_t \frac{dt}{t} \end{aligned}$$

which imply

$$dZdB = \sqrt{\frac{\langle [g_\kappa, \overline{g_\kappa}] \rangle}{2\langle [g_\kappa, \overline{g_\kappa}] \rangle + 4[g, g]}} dt.$$

Here we note the following fact. By the time change  $u = \log t$ ,  $\zeta_u := \log \eta_{e^u}$  satisfies the following SDE which is stationary in time.

$$\begin{aligned} d\zeta_u &= iC_3 du + iC_4 d\tilde{B}_u \\ \text{where } C_3 &:= -\frac{1}{\kappa} \langle |g_\kappa|^2 \rangle \in \mathbf{R}, \quad C_4 := \frac{1}{2\kappa} \sqrt{2\langle [g_\kappa, \overline{g_\kappa}] \rangle + 4[g, g]} \in \mathbf{R} \end{aligned} \tag{8.3}$$

To summarize, the following facts have been proved.

- (i) For any  $t > 0$ ,  $\eta_t$  has uniform distribution (Lemma 7.1).
- (ii) For any  $0 < t_0 < t_1 < t_2 < \dots < t_n$ , random variables  $\{\eta_{t_0}, \eta_{t_1}/\eta_{t_0}, \dots, \eta_{t_n}/\eta_{t_{n-1}}\}$  are independent (Lemma 8.3).
- (iii) For any  $t_0 > 0$ ,  $x_t = \eta_t/\eta_{t_0}$  satisfies an SDE on  $t \geq t_0$  (Lemma 8.4) :

$$dx_t = C_1 \frac{x_t}{t} dt + C_2 \frac{x_t}{\sqrt{t}} dB_t, \quad x_{t_0} = 1.$$

These facts determines (in distribution) the process  $\eta_t$  uniquely. In fact, for any  $0 < t_0 < t_1 < \dots < t_n$ , the distribution of  $\{\eta_{t_0}, \eta_{t_1}, \dots, \eta_{t_n}\}$  can be computed from that of  $\{\eta_{t_0}, \eta_{t_1}/\eta_{t_0}, \dots, \eta_{t_n}/\eta_{t_{n-1}}\}$  and the latter distribution can be determined uniquely from (ii) and (iii). Therefore the distribution of  $\{\eta_t\}$  is characterized by the constants  $C_1, C_2$ . More concretely, if we prepare

1D Brownian motion  $\{B_t\}_{t \in \mathbf{R}}$  with  $B_0 = 0$  and independent random variable  $X \in \mathbf{C}$  with uniform distribution on  $\mathbf{U}$ , a process

$$X \exp[i(C_3 u + C_4 B_u)]$$

has the same distribution as  $\{\eta_{e^u}\}$  by (8.2), (8.3).

## 9 Convergence of the joint distribution

We finish the proof of Theorem 1.3.

### 9.1 Behavior of the joint distribution

**Proposition 9.1** *For any  $c_1, \dots, c_m \in \mathbf{R}$ ,  $t > 0$ ,*

$$(\Psi_t^{(n)}(c_1), \dots, \Psi_t^{(n)}(c_m), (\theta_{nt}(\kappa))_{2\pi\mathbf{Z}}) \xrightarrow{d} (\Psi_t(c_1), \dots, \Psi_t(c_m), \phi_t), \quad (9.1)$$

*as  $n \rightarrow \infty$ , where  $(\Psi_t(c_1), \dots, \Psi_t(c_m))$  and  $\phi_t$  are independent and  $\phi_t$  is uniformly distributed on  $[0, 2\pi)$ .*

*Proof.* It suffices to show (9.1) with  $(\theta_{nt}(\kappa))_{2\pi\mathbf{Z}}$  being replaced by  $(\tilde{\theta}_{nt}(\kappa))_{2\pi\mathbf{Z}}$ , since  $(\tilde{\theta}_{nt}(\kappa))_{2\pi\mathbf{Z}}$  converges to the uniform distribution by Lemma 7.1. By Lemmas 6.8, 8.2, for any fixed  $t_0 > 0$ , the process  $\{(\Psi_t^{(n)}(c), \eta_t^{(n)})\}_{n \geq 1}$  on  $[t_0, \infty)$  is a tight family. Hence we can assume  $(\Psi_t^{(n)}(c), \eta_t^{(n)})_{t > 0} \xrightarrow{d} (\Psi_t(c), \eta_t)_{t > 0}$ . Then by Lemma 8.3  $\eta_{1/n}$  and  $\eta_t/\eta_{1/n}$  are independent.

We next consider a process  $\Psi_{n,c}(t)$  which is defined on  $[\frac{1}{n}, \infty)$  and is the solution to (6.17) with initial value  $\Psi_{n,c}(\frac{1}{n}) = \frac{c}{n}$ . [2] Proposition 4.5 proves the following fact

$$\sup_{n^{-1} < t < n} |\Psi_{n,c}(t) - \Psi(t)| \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

Since  $\eta_{\frac{1}{n}}$  and  $(\eta_t/\eta_{1/n}, \Psi_{n,c}(t))$  are independent, by letting  $n \rightarrow \infty$ , it follows that  $\eta_0 := \lim_{t \downarrow 0} \eta_t$  and  $(\eta_t/\eta_0, \Psi_t)$  are independent. Since  $\eta_0$  is uniformly distributed on  $\mathbf{U}$ ,  $\tilde{\phi}_t := \arg \eta_t = \arg \left( \eta_0 \cdot \frac{\eta_t}{\eta_0} \right)$  and  $\Psi_t$  are independent.  $\square$

## 9.2 Convergence of $\Psi_t^{(n)}$ as increasing functions

**Proposition 9.2** *Fix any  $t > 0$ . Then we can find a coupling such that the following statement is valid for a.s.*

$$\lim_{n \rightarrow \infty} (\Psi_t^{(n)})^{-1}(x) = \Psi_t^{-1}(x), \quad \lim_{n \rightarrow \infty} (2\theta_{nt}(\kappa))_{2\pi\mathbf{Z}} = \phi_t$$

for any  $x \in \mathbf{R}$  where  $\phi_t$  is uniformly distributed and independent of  $\Psi_t$ .

As is explained in section 5, Proposition 9.2 completes the proof of Theorem 1.3. To prove Proposition 9.2 we shall show below that the convergence  $\Psi_t^{(n)} \rightarrow \Psi_t$  holds in the sense of increasing function-valued process.

Let  $\mathcal{M}$  be the set of non-negative measures on  $[a, b]$ . Fix  $\{f_j\}_{j \geq 1}$  a family of smooth functions on  $[a, b]$  satisfying the property

$$\text{for } \omega \in \mathcal{M} \text{ if } \int_a^b f_j(x) d\omega(x) = 0 \text{ for any } j \geq 1 \Rightarrow \omega = 0.$$

We define a metric  $\rho$  on  $\mathcal{M}$  by

$$\rho(\omega_1, \omega_2) := \sum_{j=1}^{\infty} \frac{1}{2^j} \left( \left| \int_a^b f_j(x) d(\omega_1(x) - \omega_2(x)) \right| \wedge 1 \right).$$

Let

$$\Omega := C([0, T] \rightarrow \mathcal{M})$$

for  $T < \infty$ . We further define for a smooth function  $f$  on  $[a, b]$  a map  $\Phi_f : \Omega \rightarrow C([0, T] \rightarrow \mathbf{R})$  by

$$\begin{aligned} \Phi_f(\omega)(t) &:= \int_a^b f(x) d\omega_t(x) \\ &= [f(x)\omega_t(x)]_a^b - \int_a^b f'(x)\omega_t(x)dx. \end{aligned} \tag{9.2}$$

**Lemma 9.3** *Let  $\{\mu_n\}_{n \geq 1}$  be a family of probability measures on  $\Omega$ . Suppose for each smooth function  $f$  on  $[a, b]$  a family of probability measures  $\{\Phi_f^{-1}\mu_n\}_{n \geq 1}$  on  $C([0, T] \rightarrow \mathbf{R})$  is tight. Assume further there exists a constant  $C$  such that*

$$\mathbf{E}_{\mu_n} \left[ \sup_{0 \leq t \leq T} \int_a^b d\omega_t(x) \right] \leq C \tag{9.3}$$

holds for any  $n \geq 1$ . Then  $\{\mu_n\}_{n \geq 1}$  is tight.

*Proof.* From (9.3) we see that for any  $\epsilon > 0$ , there exists a  $M > 0$  such that

$$\mu_n \left( \sup_{0 \leq t \leq T} \int_a^b d\omega_t(x) \leq M \right) \geq 1 - \epsilon.$$

Set

$$\Omega_0 := \left\{ \omega \in \Omega \mid \sup_{0 \leq t \leq T} \int_a^b d\omega_t(x) \leq M \right\}.$$

Since  $\{\Phi_{f_j}^{-1} \mu_n\}_{n \geq 1}$  is tight for each  $j \geq 1$ , there exists a compact set  $\mathcal{K}_j$  in  $C([0, T] \rightarrow \mathbf{R})$  such that

$$\mu_n \left( \Phi_{f_j}^{-1}(\mathcal{K}_j) \right) > 1 - \frac{\epsilon}{2^j}.$$

Set

$$\mathcal{K} := \bigcap_{j=1}^{\infty} \Phi_{f_j}^{-1}(\mathcal{K}_j) \cap \Omega_0 \subset \Omega.$$

Then

$$\mu_n(\mathcal{K}^c) \leq \sum_{j=1}^{\infty} \mu_n \left( \Phi_{f_j}^{-1}(\mathcal{K}_j^c) \right) + \mu_n(\Omega_0^c) \leq \epsilon + \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = 2\epsilon. \quad (9.4)$$

We show  $\mathcal{K}$  is compact in  $\Omega$ . Let  $\{\omega_n\}_{n \geq 1}$  be a sequence in  $\mathcal{K}$ . Since  $\mathcal{K}_1$  is compact, there exists a subsequence  $\{n_i^1\}$  along which  $\Phi_{f_1}(\omega_{n_i^1})$  is uniformly convergent in  $C([0, T] \rightarrow \mathbf{R})$ . Then, using the compactness of  $\mathcal{K}_2$  we can find a subsequence  $\{n_i^2\}$  of  $\{n_i^1\}$  along which  $\Phi_{f_2}(\omega_{n_i^2})$  is uniformly convergent in  $C([0, T] \rightarrow \mathbf{R})$ . Continuing this procedure for each  $j$  we find a subsequence  $\{n_i^j\}$  of  $\{n_i^{j-1}\}$  along which  $\Phi_{f_j}(\omega_{n_i^j})$  is uniformly convergent in  $C([0, T] \rightarrow \mathbf{R})$ . Let  $m_i = n_i^i$ . Then for each  $j \geq 1$ ,  $\Phi_{f_j}(\omega_{m_i})$  converges uniformly in  $C([0, T] \rightarrow \mathbf{R})$ . Since, for any  $f \in C[a, b]$  and  $\epsilon' > 0$ , there exists a finite linear combination  $g$  of  $\{f_j\}$  such that

$$\sup_{x \in [a, b]} |f(x) - g(x)| < \epsilon'.$$

We easily have

$$\sup_{t \in [0, T]} |\Phi_f(\omega_{m_i})(t) - \Phi_g(\omega_{m_i})(t)| \leq \epsilon' M$$

where we have used  $\int_a^b d\omega_t(x) \leq M$  for any  $\omega \in \mathcal{K}$ . Therefore we see that the limit

$$\lim_{i \rightarrow \infty} \Phi_f(\omega_{m_i})(t)$$

exists uniformly w.r.t.  $t \in [0, T]$ , which implies that there exists a  $\omega \in \Omega$  satisfying

$$\int_a^b d\omega_t(x) \leq M \text{ and } \lim_{i \rightarrow \infty} \Phi_f(\omega_{m_i}) = \Phi_f(\omega)(t)$$

for any  $t \in [0, T]$  and  $f \in C([a, b])$ . Consequently we have the compactness of  $\mathcal{K}$  which together with (9.4) shows the tightness of  $\{\mu_n\}_{n \geq 1}$ .  $\square$

We would like to check that the conditions for Lemma 9.3 are satisfied for  $\Psi_t^{(n)}(\cdot)$ . The inequality (9.3) follows from Lemma 6.6. In view of (9.2), the required tightness is implied by the following lemma.

**Lemma 9.4** *For  $f \in C^\infty(a, b)$  let*

$$g_n(t) := \int_a^b f(x) \Psi_t^{(n)}(x) dx.$$

*Then, as a family of probability measures on  $C([0, T] \rightarrow \mathbf{R})$ ,  $\{g_n\}_{n \geq 1}$  is tight.*

*Proof.* It is sufficient to show that following two equations.

$$(1) \lim_{A \rightarrow \infty} \sup_n \mathbf{P}(|g_n(0)| \geq A) = 0,$$

$$(2) \text{ For any } \rho > 0,$$

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbf{P} \left( \sup_{|t-s| < \delta} |g_n(t) - g_n(s)| > \rho \right) = 0.$$

By Lemma 6.6, (1) is clear. By bounding  $f$ , the following equation implies (2).

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbf{P} \left[ \sup_{|t-s| < \delta} \int_a^b \left| \Psi_t^{(n)}(x) - \Psi_s^{(n)}(x) \right| dx > \rho \right] = 0.$$

Here we borrow an argument in [2] Proposition 2.5 : We divide  $[a, b]$  into  $N$ -intervals

$$x_j = a + \frac{b-a}{N}x_j, \quad j = 0, 1, \dots, N-1,$$

and have

$$\int_a^b |\Psi_t^{(n)}(x) - \Psi_s^{(n)}(x)| dx = \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} |\Psi_t^{(n)}(x) - \Psi_s^{(n)}(x)| dx. \quad (9.5)$$

Since  $\Psi_t^{(n)}(x)$  is increasing with respect to  $x$ , for  $x \in [x_j, x_{j+1}]$  the integrand is bounded from above by

$$\begin{aligned} & |\Psi_t^{(n)}(x) - \Psi_s^{(n)}(x)| \\ & \leq \Psi_t^{(n)}(x_{j+1}) - \Psi_t^{(n)}(x_j) + |\Psi_t^{(n)}(x_j) - \Psi_s^{(n)}(x_j)| + \Psi_s^{(n)}(x_{j+1}) - \Psi_s^{(n)}(x_j). \end{aligned}$$

Substituting it into (9.5) yields

$$\begin{aligned} J &:= \int_a^b |\Psi_t^{(n)}(x) - \Psi_s^{(n)}(x)| dx \\ &\leq \sum_{j=0}^{N-1} \frac{1}{N} (b-a) \left( \Psi_t^{(n)}(x_{j+1}) - \Psi_t^{(n)}(x_j) \right) + (t \leftrightarrow s) \\ &\quad + \sum_{j=0}^N \frac{1}{N} (b-a) |\Psi_t^{(n)}(x_j) - \Psi_s^{(n)}(x_j)| \\ &= \frac{1}{N} (b-a) \left( \Psi_t^{(n)}(b) - \Psi_t^{(n)}(a) \right) + (t \leftrightarrow s) \\ &\quad + \sum_{j=0}^N \frac{1}{N} (b-a) |\Psi_t^{(n)}(x_j) - \Psi_s^{(n)}(x_j)| \\ &=: I + II. \end{aligned}$$

Thus we decompose the probability in question into two terms.

$$\begin{aligned} \mathbf{P} \left( \sup_{|t-s| < \delta} J > \rho \right) &\leq \mathbf{P} \left( \sup_{|t-s| < \delta} I > \rho/2 \right) + \mathbf{P} \left( \sup_{|t-s| < \delta} II > \rho/2 \right) \\ &=: III + IV. \end{aligned}$$

The  $III$ -term can be estimated by Lemma 6.6.

$$\begin{aligned}
III &\leq \mathbf{P} \left( \frac{b-a}{N} \left( \Psi_t^{(n)}(b) - \Psi_t^{(n)}(a) \right) > \rho/4 \right) + (t \leftrightarrow s) \\
&\leq \frac{4}{\rho} \cdot \frac{b-a}{N} \mathbf{E} \left[ \left( \Psi_t^{(n)}(b) - \Psi_t^{(n)}(a) \right) \right] + (t \leftrightarrow s) \\
&\leq 2 \cdot \frac{4}{\rho} \cdot \frac{b-a}{N} \mathbf{E} \left[ \sup_{0 \leq t \leq T} \left( \Psi_t^{(n)}(b) \right) \right] \leq \frac{C}{N}.
\end{aligned}$$

Thus for any  $\epsilon > 0$  we take  $N$  large enough independently of  $\delta$  to have

$$III < \frac{\epsilon}{2}.$$

For such fixed  $N$ , we have

$$\begin{aligned}
IV &\leq \sum_{j=0}^N \mathbf{P} \left( \frac{b-a}{N} \sup_{|t-s| < \delta} |\Psi_t^{(n)}(x_j) - \Psi_s^{(n)}(x_j)| > \frac{\rho}{2N} \right) \\
&= \sum_{j=0}^N \mathbf{P} \left( \sup_{|t-s| < \delta} |\Psi_t^{(n)}(x_j) - \Psi_s^{(n)}(x_j)| > \frac{\rho}{2(b-a)} \right).
\end{aligned}$$

Since  $\{\Psi_t^{(n)}(x_j)\}_{j=0}^N$  is tight by Lemma 6.8, we can let  $IV < \epsilon/2$  by taking  $n$  large and then taking  $\delta > 0$  small.  $\square$

We identify an element of  $\mathcal{M}$  with a non-decreasing and right continuous function  $\omega$  on  $[a, b]$  satisfying  $\omega(a) = 0$ . Then  $\omega_n$  converges to  $\omega \in \Omega$  if and only if  $\omega_n(x) \rightarrow \omega(x)$  at any point of continuity of  $\omega$ .

**Lemma 9.5** *Suppose  $\{\omega_n\}_{n \geq 1} \subset \mathcal{M}$  converges to  $\omega$  of  $\mathcal{M}$ . Assume  $\omega$  is continuous. Then the convergence is uniform.*

*Proof.* Assume  $\{\omega_n\}_{n \geq 1}$  does not converge to  $\omega$  uniformly. Then there exists a sequence  $n_1 < n_2 < \dots$ ,  $\{t_k\}_{k \geq 1}$  and a positive number  $\epsilon_0$  such that

$$|\omega_{n_k}(t_k) - \omega(t_k)| \geq \epsilon_0 \tag{9.6}$$

is valid for any  $k = 1, 2, \dots$ . We can assume  $t_k \rightarrow t_0 \in [a, b]$  keeping  $t_1 < t_2 < \dots < t_0$ . Then

$$\omega_{n_k}(t_l) - \omega(t_k) \leq \omega_{n_k}(t_k) - \omega(t_k) \leq \omega_{n_k}(t_0) - \omega(t_k)$$

for any  $l < k$ , hence letting  $k \rightarrow \infty$ , we have

$$\begin{aligned}\omega(t_l) - \omega(t_0) &\leq \liminf_{k \rightarrow \infty} (\omega_{n_k}(t_k) - \omega(t_k)) \\ &\leq \limsup_{k \rightarrow \infty} (\omega_{n_k}(t_k) - \omega(t_k)) \leq \omega(t_0) - \omega(t_0) = 0.\end{aligned}$$

Consequently, letting  $l \rightarrow \infty$ , we see

$$\lim_{k \rightarrow \infty} (\omega_{n_k}(t_k) - \omega(t_k)) = 0,$$

which contradicts (9.6).

### *Proof of Proposition 9.2*

By Lemma 9.3, the sequence of increasing function-valued process  $\{\Psi_t^{(n)}(\cdot)\}_n$  is tight. Hence  $(\Psi_t^{(n_k)}, (2\theta_{n_k t})_{2\pi\mathbf{Z}}) \xrightarrow{d} (\Psi_t, \phi_t)$  for some subsequence  $\{n_k\}$ . By Skorohod's theorem, we can suppose  $(\Psi_t^{(n_k)}, (2\theta_{n_k t})_{2\pi\mathbf{Z}}) \xrightarrow{a.s.} (\Psi_t, \phi_t)$ . Hence in particular we fix any  $t > 0$  and obtain

$$\rho(\Psi_t^{(n_k)}, \Psi_t) = \sum_{j \geq 1} \frac{1}{2^j} \left( \left| \int_a^b f_j(x) d(\Psi_t^{(n_k)}(x) - \Psi_t(x)) \right| \wedge 1 \right) \xrightarrow{n \rightarrow \infty} 0, \quad a.s.$$

By Lemma 6.11  $\Psi_t$  is continuous and increasing. Hence for a.s.,  $\Psi_t^{(n_k)}(x) \rightarrow \Psi_t(x)$  holds for any  $x$ . Moreover by Lemma 3.3  $(\Psi_t^{(n_k)})^{-1}(x) \xrightarrow{a.s.} \Psi_t^{-1}(x)$ . Therefore Proposition 9.2 is proved.  $\square$

**Acknowledgement** This work is partially supported by JSPS grants Kiban-C no.22540163(S.K.) and no.22540140(F.N.).

## References

- [1] Avila, A., Last, Y., and Simon, B., : Bulk Universality and Clock Spacing of zeros for Ergodic Jacobi Matrices with A.C. spectrum, *Anal. PDE* **3**(2010), no.1, 81-108.
- [2] Killip, R., Stoiciu, M., : Eigenvalue statistics for CMV matrices : from Poisson to clock via random matrix ensembles, *Duke Math.* **146**, no. 3(2009),
- [3] Kotani, S. Ushiroya, N.: One-dimensional Schrödinger operators with random decaying potentials, *Comm. Math. Phys.* **115**(1988), 247-266.



- [4] Kotani, S. : On limit behavior of eigenvalues spacing for 1-D random Schrödinger operators, Kokyuroku Bessatsu **B27**(2011), 67-78.
- [5] Kritchevski, E., Valkó B., Virág, B., : The scaling limit of the critical one-dimensional random Schrödinger operators, arXiv:1107.3058.
- [6] Molchanov, S. A. : The local structure of the spectrum of the one-dimensional Schrödinger operator, Comm. Math. Phys. **78**(1981), 429-446.
- [7] Minami, N., : Local fluctuation of the spectrum of a multidimensional Anderson tight-binding model, Comm. Math. Phys. **177**(1996), 709-725.